TENSOR PRODUCTS OF ABELIAN GROUPS

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1. **Introduction.** Let G and H be Abelian groups. Their direct sum $G \oplus H$ consists of all pairs (g, h), with $(g, h) + (g', h') = (g + g', h + h')$. If we consider G and H as subgroups of $G \oplus H$, with elements $g = (g, 0)$ and $h = (0, h)$, then we may form $g + h$, and the ordinary laws of *addition* hold. Our object in this paper is to construct a group $G \circ H$ from G and H, in which we can form $g \cdot h$, with the properties of *multiplication*; that is, the distributive laws

$$(1.1) \qquad (g + g')\cdot h = g\cdot h + g'\cdot h, \qquad g\cdot(h + h') = g\cdot h + g\cdot h'$$

hold. Clearly $G \circ H$ must contain elements of the form $\sum g_i\cdot h_i$; it turns out (Theorem 1) that with these elements, *assuming only* (1.1), *we obtain an Abelian group*, which we shall call the *tensor product* of G and H.[1]

The tensor product is known in one important case; namely, in tensor analysis (see §4, (b), and §11), though the definition in the form here given does not seem to have been used. Certain other cases are known (see §4). We refer to the examples there given for further indications of the scope of the theory. A direct product of algebras has been constructed by J. L. Dorroh,[2] by methods closely allied to those of the present paper.

As is to be expected, we see in Part I that when we multiply several groups together, the associative and commutative laws hold; also the distributive laws with respect to direct sums and difference groups. The group of integers plays the rôle of a unit group.[3] The rest of Part I is devoted largely to a study of the relation between groups with operator rings and tensor products; in particular, divisibility properties are considered.

In Part II, a detailed study of tensor products of linear spaces is made; we now assume $rg\cdot h = g\cdot rh$ (r real). With any element α of $G \circ H$ are associated subspaces $G(\alpha)$ of G and $H(\alpha)$ of H; their dimensions equal the minimum number of terms in an expression $\sum g_i\cdot h_i$ for α, and in this expression the g_i and h_i form bases in $G(\alpha)$ and $H(\alpha)$. The elementary operations of tensor algebra are derived, and a direct manner of introducing covariant differentiation is indicated.[4] If the linear spaces are topological, a topology may be introduced into

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[1] This is so even if G and H are not Abelian; see Theorem 11. If G and H are linear or topological, we use a slightly different definition.

[2] J. L. Dorroh, *Concerning the direct product of algebras*, Annals of Mathematics, vol. 36 (1935), pp. 882–885. The author is indebted to the referee for pointing out this paper to him.

[3] In linear spaces, the group of real numbers also is a unit.

[4] Some of these results have been derived independently by H. E. Robbins.

the tensor product. If the spaces are not of finite dimension, there are of course various topologies possible in the product; the one we give is probably at an extreme end, in that a neighborhood of 0 in any topology will contain a neighborhood of the sort here given. The topology has certain defects in that the associative and distributive laws seem not to hold in general with topology preserved. In the case of Hilbert spaces, there is a natural definition of the topology in the product (see Murray and von Neumann, reference in §4, (c)). In the intermediate case of Banach spaces, probably the norm $| \alpha |$ may be defined as the lower bound of numbers $\sum | g_i | | h_i |$ for expressions $\sum g_i \cdot h_i$ of α.[5]

In topological groups which contain denumerable dense sets, the product may be given a topology, as is shown in Part III; it agrees with that in Part II when both are defined. Again, in complicated groups, other topologies are possible and perhaps preferable. Finally, for a more complete theory, one must allow infinite sums $\sum g_i \cdot h_i$.

2. **Notations.** Write $G \approx H$ if G and H are isomorphic. The symbol 0 means the zero in any group, or the group with only the zero element. $A \cap B$ is the set of elements in both A and B. ag (a an integer > 0) means $g + \cdots + g$ (a terms); $(-a)g = a(-g)$, $0g = 0$. $g + A$ is the set of all $g + g'$, g' in A; similarly for $A + B$. $g \cdot B$ is the set of all $g \cdot h$, h in B, etc. $aA =$ all ag, g in A. Note that $2A \subset A + A$, etc. Write $a \mid g$ if there is a g' with $ag' = g$; g is then "divisible" by the integer a. $a \mid A$ means $a \mid g$ for all g in A. G is "completely divisible" if for every $a \neq 0$, $a \mid G$, i.e., $aG = G$. The "nullifier" of H in G (of G in H) is the set of all g (all h) such that $g \cdot h = 0$ for all h in H (all g in G).

Let $\sum^* A$ denote the set of all finite sums $a_1 + \cdots + a_k$, a_i in A, any k; this is a subgroup of G (if $A \subset G$). $\sum_i^* A_i$ is the set of all $a_1 + \cdots + a_k$ (a_i in A_i, any k).

Let $G \oplus H$ and $G \ominus G'$ denote direct sums and difference groups. There is a "natural homomorphism" of G into $G \ominus G'$. Some particular groups we shall use are: I_0 = group of integers; $I_\mu = I_0 \ominus \mu I_0$ = integers mod μ (with elements a_μ for integral a); Rt = rational numbers; Rl = real numbers. Set $G_\mu = G \ominus \mu G$.

I. Discrete groups

3. **Discrete tensor products.** Let G and H be groups (not necessarily Abelian), with the operation $+$. Let $\mathfrak{S}$ be the set of all symbols

$$(g_1, h_1; \cdots; g_n, h_n) \qquad (g_i \text{ in } G,\ h_i \text{ in } H,\ n \text{ any integer} > 0).$$

We add two symbols by the rule

$$(g_1, h_1; \cdots) + (g_{n+1}, h_{n+1}; \cdots) = (g_1, h_1; \cdots; g_{n+1}, h_{n+1}; \cdots).$$

[5] This definition was suggested to me by H. E. Robbins.

Clearly $+$ is associative. We may put any element of $\mathfrak{S}$ in the normal form $(g_1, h_1) + \cdots + (g_n, h_n)$; if we write

$$g_i \times h_i = (g_i, h_i),$$

we obtain

$$(g_1, h_1; \cdots; g_n, h_n) = g_1 \times h_1 + \cdots + g_n \times h_n.$$

Define two equivalence relations as follows:

$$\cdots + (g + g') \times h + \cdots \sim \cdots + g \times h + g' \times h + \cdots, \tag{3.1}$$

$$\cdots + g \times (h + h') + \cdots \sim \cdots + g \times h + g \times h' + \cdots. \tag{3.2}$$

Any succession $s_1 \sim s_2 \sim \cdots \sim s_p$ we shall call an *equivalence sequence* between s_1 and s_p. If two elements s, s' are joined by an equivalence sequence, we say they are *equivalent*, $s \sim s'$. Let also $s \sim s$. The elements of $\mathfrak{S}$ fall into subsets under this relation; these form the elements of the *discrete tensor product* $G \circ H$. In case G and H are discrete, we call this the *tensor product*, in agreement with the definition in Part III. Let $\sum g_i \cdot h_i = g_1 \cdot h_1 + \cdots$ be the element of $G \circ H$ containing the element $\sum g_i \times h_i$ of $\mathfrak{S}$.

To define the group operation, which we temporarily call $\oplus$, in $G \circ H$, take any α and α', and let $\sum g_i \times h_i$ and $\sum g_i' \times h_i'$ be corresponding elements of $\mathfrak{S}$; we set

$$\alpha \oplus \alpha' = \sum g_i \cdot h_i + \sum g_i' \cdot h_i'. \tag{3.3}$$

We must show that this is independent of the choices of $s = \sum g_i \times h_i$ and $s' = \sum g_i' \times h_i'$. If we had chosen t and t', then there are equivalence sequences joining s to t and s' to t'; applying these sequences to $\sum g_i \times h_i + \sum g_i' \times h_i'$ shows that the same element $\alpha \oplus \alpha'$ is determined. Henceforth we use $+$ instead of $\oplus$. Note that $+$ is associative, and (1.1) holds.

We prove in succession the following facts.

(a)
$$\begin{aligned} g \cdot 0 = (g + g - g) \cdot 0 &= g \cdot 0 + g \cdot 0 + (-g) \cdot 0 = g \cdot (0 + 0) + (-g) \cdot 0 \\ &= g \cdot 0 + (-g) \cdot 0 = (g - g) \cdot 0 = 0 \cdot 0; \end{aligned}$$

similarly $0 \cdot h = 0 \cdot 0$.

(b)
$$g \cdot h + 0 \cdot 0 = g \cdot h + g \cdot 0 = g \cdot (h + 0) = g \cdot h,$$

and hence $0 \cdot 0 = g \cdot 0 = 0 \cdot h$ plays the rôle of the identity.

(c)
$$\begin{aligned} g \cdot h = g \cdot h + 0 \cdot (-h) &= g \cdot h + g \cdot (-h) + (-g) \cdot (-h) \\ &= g \cdot 0 + (-g) \cdot (-h) = (-g) \cdot (-h). \end{aligned}$$

Next, we may operate with the product as if G and H were Abelian. For

(d)
$$\begin{aligned} g \cdot (h + h') = g \cdot h + g \cdot h' &= (-g) \cdot (-h) + (-g) \cdot (-h') \\ &= (-g) \cdot (-h - h') = g \cdot (h' + h); \end{aligned}$$

similarly $(g + g')\cdot h = (g' + g)\cdot h$. Also

$$g\cdot(h + h' + h'') = g\cdot h + g\cdot(h' + h'') = g\cdot h + g\cdot(h'' + h') = g\cdot(h + h'' + h'), \text{ etc.}$$

Finally, the operation in $G \circ H$ is commutative. For [6]

$$\alpha = (g + g')\cdot(h' + h) = g\cdot(h' + h) + g'\cdot(h' + h) = g\cdot h' + g\cdot h + g'\cdot h' + g'\cdot h,$$

also

$$\alpha = (g + g')\cdot h' + (g + g')\cdot h = g\cdot h' + g'\cdot h' + g\cdot h + g'\cdot h,$$

and hence

$$\text{(e)} \qquad g\cdot h + g'\cdot h' = (-g)\cdot h' + \alpha + (-g')\cdot h = g'\cdot h' + g\cdot h.$$

Remark. We would have obtained the same results if we had replaced the elementary equivalence relations by

$$\cdots + (g + g') \times h + \cdots \sim \cdots + g' \times h + g \times h + \cdots, \text{ etc.}$$

THEOREM 1. *$G \circ H$ is an Abelian group; the identity is $0\cdot 0 = g\cdot 0 = 0\cdot h$, and the inverse of $g\cdot h$ is*

$$(3.4) \qquad -(g\cdot h) = (-g)\cdot h = g\cdot(-h).$$

The distributive laws (1.1) *hold.*

This follows from the above results. Because of (d), *we henceforth assume G and H are Abelian,* except in Theorem 11.

THEOREM 2. *In any $G \circ H$, for any integer a,*

$$(3.5) \qquad a(g\cdot h) = ag\cdot h = g\cdot ah.$$

For instance,

$$(-2)g\cdot h = (-g - g)\cdot h = -[(g + g)\cdot h] = -[g\cdot h + g\cdot h] = (-2)(g\cdot h).$$

Using the distributive laws, we may use summation signs as usual; for instance,

$$\sum_i \left(\sum_j a_{ij}g_j\right)\cdot h_i = \sum_i \sum_j (a_{ij}g_j\cdot h_i) = \sum_j \sum_i (g_j\cdot a_{ij}h_i) = \sum_j \left(g_j\cdot \sum_i a_{ij}h_i\right).$$

4. **Examples.** A system with both "addition" and "multiplication" may in general be defined by starting with a system or systems, using addition alone,

[6] For a direct proof, we have

$$g\cdot h + g'\cdot h' = g\cdot h + g\cdot h' + (-g + g')\cdot h' = g\cdot(h + h') + (g' - g)\cdot(h + h') + (g' - g)\cdot(-h) = (g + g' - g)\cdot(h + h') + g'\cdot(-h) + (-g)\cdot(-h) = g'\cdot(h + h' - h) + g\cdot h = g'\cdot h' + g\cdot h.$$

forming a tensor product, and defining new equality relations. Specifically, any group pair is an example.

(a) The Abelian groups G and H form a *group pair* with respect to the group Z if a multiplication $g \times h = z$ is given, satisfying both distributive laws. *Any such group pair may be defined by choosing a homomorphism of $G \circ H$ into Z.* Clearly

$$\phi(\sum g_i \cdot h_i) = \sum g_i \times h_i$$

has the required properties. Practically all further examples come under this head.

(b) The most important example of a true tensor product (and the example from which we chose the word "tensor") is probably the following. If G is the tangent vector space at a point of a differentiable manifold, then $G \circ G$ is the space of contravariant tensors of order 2 at the point. (Here $G \circ G$ is not the discrete, but the reduced, or topological, tensor product; see Part II or Part III. The same remark applies to other examples below.) Using also the "conjugate space" $L(G)$ and iterated tensor products gives tensors of all orders (see §11). Of course these spaces are usually defined in terms of coördinate systems in G.

Note that in terms of a fixed coördinate system, $G \circ G$ gives: vector times vector equals matrix. For a generalization, see (i) below.

(c) If G in (b) is replaced by Hilbert space, $G \circ G$ is a Hilbert space,[7] except for the completeness postulate (which could be obtained by completing the space or allowing certain infinite sums in $G \circ G$).

(d) The true tensor product $G \circ H$ has also been used in case one of G, H has a finite number of generators, and has been applied in topology.[8] From the examples (j) and Theorems 3 and 5 below, we may at once determine $G \circ H$ if both G and H have finite sets of generators.

The remaining examples are in general not true tensor products, but come under the heading (a). The general case $G \circ H \to Z$ does not often occur. The case $G \circ G \to Z$ appears in (b). The cases $G \circ H \to H$ and $G \circ G \to G$ appear in (e) and (g) below.

(e) If G is a group, with "operators" from the group R, i.e., $r \cdot g = g'$, the distributive laws are generally assumed; we have $R \circ G \to G$. Here one generally lets R be a ring (see §6).

(f) If G is a group and R is a ring, and we wish to form from G a group G^*

[7] See F. J. Murray and J. von Neumann, *On rings of operators*, Annals of Mathematics, vol. 37(1936), pp. 116–229, Chapter I. As a bounded operator A in G corresponds uniquely to an element f in G: $A(g) = (f, g)$, their space $G \otimes G$ corresponds to our $G \circ G$. M. H. Stone and J. W. Calkin have also considered a direct definition of $G \circ G$ such as we give. Compare also M. Kerner, *Abstract differential geometry*, Compositio Mathematica, vol. 4 (1937), pp. 308–341.

[8] See Alexandroff-Hopf, *Topologie* I, pp. 585–586 and p. 233, (15), and H. Freudenthal, Fundamenta Mathematicae, vol. 29(1937). The definition of $G \circ H$ is indirect. The case that one of G, H is a free group has been studied by H. Freudenthal, Compositio Mathematica, vol. 4(1937), pp. 145–234, Chapter III.

which "admits" R as operator ring, we need merely use $G^* = R \circ G$ (see Theorem 12 below). If we wish to replace G by a group G^* in which division by any integer $\neq 0$ is possible and unique, we use $G^* = Rt \circ G$ (see §8).

(g) If G is a group, any choice of $G \circ G \rightarrow G$ makes G a ring (in general non-associative), and conversely.

(h) Let V_p, V_q and V_r be linear spaces (= vector spaces) of dimensions p, q and r. Set $G = Ch_{V_q}(V_p)$ (= group of linear maps of V_p into V_q), $H = Ch_{V_r}(V_q)$, $Z = Ch_{V_r}(V_p)$. Obviously, we have $G \circ H \rightarrow Z$. G, H, Z, and $G \circ H$ are vector spaces of dimensions pq, qr, pr, and pq^2r. Hence $Z \approx G \circ H$ is possible only if $q = 1$, i.e., $V_q \approx Rl$. In this case it is true, as shown by (10.7) and (10.11) below. If we choose fixed coördinate systems in V_p, V_q and V_r, then G, H and Z may be interpreted as groups of matrices.

(i) If $G = H$ is the (additive) group of continuous functions $g(x)$, $0 \leqq x \leqq 1$, we may interpret $G \circ H$ as a subgroup of the group of continuous functions $z(x, y)$, $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, with $g \cdot h$ corresponding to $z(x, y) = g(x)h(y)$. As is well known from the theory of integral equations, if we allow infinite sums, we may obtain all continuous functions $z(x, y)$.

(j) Finally, we give some examples of tensor products, using the groups most commonly used as coefficient groups in topology. Let Rt_1 and Rl_1 be Rt and Rl reduced mod 1.

$$I_0 \circ G \approx G, \qquad I_\mu \circ G \approx G_\mu \qquad \text{(Theorems 7, 8)},$$
$$I_\mu \circ I_\nu \approx I_{(\mu,\nu)},$$
$$I_\mu \circ Rt \approx I_\mu \circ Rl \approx I_\mu \circ Rt_1 \approx I_\mu \circ Rl_1 \approx 0 \qquad (\mu > 0),$$
$$Rt \circ Rt \approx Rt, \qquad Rt \circ Rl \approx Rl \circ Rl \approx Rl,$$
$$Rt \circ Rt_1 \approx Rt \circ Rl_1 \approx Rt_1 \circ Rt_1, \text{ etc.}, \approx 0.$$

5. **General properties.** We first consider commutative and associative properties.

THEOREM 3. *There is a natural isomorphism $G \circ H \approx H \circ G$, given by*

$$\phi(\textstyle\sum g_i \cdot h_i) = \sum h_i \cdot g_i . \tag{5.1}$$

THEOREM 4. *There are natural isomorphisms*

$$F \circ (G \circ H) \approx F \circ G \circ H \approx (F \circ G) \circ H,$$

where $F \circ G \circ H$ is the group of all $\sum f_i \cdot g_i \cdot h_i$, using the three distributive laws. The isomorphisms are given by

$$\phi(\textstyle\sum f_i \cdot g_i \cdot h_i) = \sum (f_i \cdot g_i) \cdot h_i , \qquad \psi(\sum f_i \cdot g_i \cdot h_i) = \sum f_i \cdot (g_i \cdot h_i). \tag{5.2}$$

The first theorem is evident; we prove the second, using ϕ. The definition of ϕ is unique, as any equivalence relation in the $\sum f_i \cdot g_i \cdot h_i$ corresponds to one in the $\sum (f_i \cdot g_i) \cdot h_i$. If $\phi(\sum f_i \cdot g_i \cdot h_i) = 0$, then an equivalence sequence carries

$\sum(f_i \cdot g_i) \cdot h_i$ into 0; a corresponding sequence carries $\sum f_i \cdot g_i \cdot h_i$ into 0; hence ϕ is an isomorphism into a subgroup of $(F \circ G) \circ H$. Finally, given any

$$\sum_i z_i \cdot h_i = \sum_i \left(\sum_j f_{ij} \cdot g_{ij}\right) \cdot h_i = \sum_{i,j} (f_{ij} \cdot g_{ij}) \cdot h_j$$

in $(F \circ G) \circ H$, ϕ carries $\sum f_{ij} \cdot g_{ij} \cdot h_i$ into it. This completes the proof.

Next we prove the distributive laws with respect to direct sums and difference groups.

THEOREM 5. *There is a natural isomorphism*

$$(F \oplus G) \circ H \approx F \circ H \oplus G \circ H,$$

given by

$$(5.3)\quad \phi[(f_1, g_1) \cdot h_1 + \cdots + (f_n, g_n) \cdot h_n] = (f_1 \cdot h_1 + \cdots + f_n \cdot h_n, g_1 \cdot h_1 + \cdots + g_n \cdot h_n).$$

To show that ϕ is uniquely defined, we have, for instance, as $(f, g) + (f', g') = (f + f', g + g')$,

$$\begin{aligned}\phi[\cdots + (f, g) \cdot h + (f', g') \cdot h + \cdots] &= (\cdots + f \cdot h + f' \cdot h + \cdots, \cdots + g \cdot h + g' \cdot h + \cdots) \\ &= (\cdots + (f + f') \cdot h + \cdots, \cdots + (g + g') \cdot h + \cdots) \\ &= \phi[\cdots + \{(f, g) + (f', g')\} \cdot h + \cdots].\end{aligned}$$

ϕ maps the first group into the whole of the second; for

$$(5.4)\quad \phi[(f_1, 0) \cdot h_1 + \cdots + (0, g_1) \cdot h_1' + \cdots] = (f_1 \cdot h_1 + \cdots, g_1 \cdot h_1' + \cdots).$$

Clearly ϕ is a homomorphism. Now suppose $\phi(\alpha) = 0$; let α be given as in (5.3). First, we may transform α into the form of the left side of (5.4). For each half of the right side of (5.3), there is an equivalence sequence carrying it into 0. There are corresponding sequences acting on the left side of (5.4), which shows that $\alpha = 0$. Hence ϕ is an isomorphism.

THEOREM 6. *If G' is a subgroup of G, there is a natural isomorphism*

$$(G \ominus G') \circ H \approx G \circ H \ominus \sum{}^* (G' \cdot H),$$

given as follows. If ψ and Ψ are the natural homomorphisms of G into $G \ominus G'$ and of $G \circ H$ into $G \circ H \ominus \sum^*(G' \cdot H)$, we set

$$(5.5)\quad \phi[\psi(g_1) \cdot h_1 + \cdots + \psi(g_n) \cdot h_n] = \Psi(g_1 \cdot h_1 + \cdots + g_n \cdot h_n).$$

By Theorem 3, there is a similar relation with G and H interchanged.

To show that ϕ is uniquely defined, suppose first that $\psi(g_1) = \psi(\bar{g}_1)$. Then $\bar{g}_1 = g_1 + g'$ (g' in G'), and

$$\Psi(\bar{g}_1 \cdot h_1 + \cdots) = \Psi(g_1 \cdot h_1 + \cdots) + \Psi(g' \cdot h_1) = \Psi(g_1 \cdot h_1 + \cdots).$$

The rest of the proof is like previous proofs. For instance, if the element (5.5) vanishes, then $\sum g_i \cdot h_i$ is in $\sum^*(G' \cdot H)$, and hence may be transformed into the form $\sum g_i' \cdot h_i'$ (g_i' in G'). The same transformations may be carried out on the left side of (5.5); as $\psi(g_i') = 0$, this gives $\sum \psi(g_i) \cdot h_i = 0$.

Remark. $\sum^*(G' \cdot H)$ is perhaps "smaller" than $G' \circ H$; for instance, if $G = I_0$, $G' = 2G$, $H = I_2$, then $G' \circ H \approx I_2$, $\sum^*(G' \cdot H) \approx 0$. But *there is a natural homomorphism of $G' \circ H$ onto the whole of $\sum^*(G' \cdot H)$*, clearly. Compare Theorem 28, Part II.

THEOREM 7. *There is a natural isomorphism $I_0 \circ G \approx G$, given by*

$$\phi(\sum a_i \cdot g_i) = \sum a_i g_i . \tag{5.6}$$

The proof is like previous proofs. Note that we have a normal form for elements of $I_0 \circ G$: if we use Theorem 2,

$$\sum a_i \cdot g_i = \sum 1 \cdot a_i g_i = 1 \cdot \sum a_i g_i = 1 \cdot g'. \tag{5.7}$$

The expression of an element in the normal form is unique, by the theorem.

THEOREM 8. *There is a natural isomorphism $I_\mu \circ G \approx G_\mu$, given by*[9]

$$\phi(\sum_i a_\mu^i \cdot g^i) = \sum_i a^i g_\mu^i. \tag{5.8}$$

Using Theorems 6 and 7, we see easily that the following isomorphism is the one given by the theorem:

$$I_\mu \circ G = (I_0 \ominus \mu I_0) \circ G \approx I_0 \circ G \ominus \sum{}^* (\mu I_0 \cdot G)$$
$$= I_0 \circ G \ominus \sum{}^* (I_0 \cdot \mu G) \approx I_0 \circ (G \ominus \mu G) \approx G_\mu .$$

THEOREM 9. *If G is completely divisible and every element of H is of finite order, then $G \circ H \approx 0$.*

For if $mh = 0$, then $g \cdot h = mg' \cdot h = g' \cdot mh = 0$.

THEOREM 10. *If G' and H' are subgroups of the nullifiers of H and G in G and H, respectively, then there are natural isomorphisms*

$$G \circ H \approx (G \ominus G') \circ H \approx G \circ (H' \ominus H) \approx (G \ominus G') \circ (H \ominus H');$$

if ϕ and ψ are the natural isomorphisms of G into $G \ominus G'$ and of H into $H \ominus H'$, these are given by

$$\sum g_i \cdot h_i \approx \sum \phi(g_i) \cdot h_i \approx \sum g_i \cdot \psi(h_i) \approx \sum \phi(g_i) \cdot \psi(h_i).$$

First, applying Theorem 6, we find, as $G' \cdot H = 0$,

$$G \circ H \approx G \circ H \ominus \sum{}^* (G' \cdot H) \approx (G \ominus G') \circ H, \text{ etc.}$$

Next, for any h' in H', $\phi(g) \cdot h'$ corresponds to $g \cdot h' = 0$ in the first isomorphism above; hence $(G \ominus G') \cdot H' = 0$, and

$$(G \ominus G') \circ H \approx (G \ominus G') \circ H \ominus \sum{}^* ((G \ominus G') \cdot H') \approx (G \ominus G') \circ (H \ominus H').$$

[9] g_μ is the element of G_μ corresponding to g in G.

We end by showing that the discrete tensor product of any two groups, not necessarily Abelian, is isomorphic to the discrete tensor product of the two groups "made Abelian".

THEOREM 11. *Let G and H be any two groups, and let G' and H' be their commutator subgroups. Then there is a natural isomorphism*

$$G \circ H \approx (G \ominus G') \circ (H \ominus H').$$

Because of Theorem 10, we need merely show that any commutator is in the nullifier of the other group; this follows at once from §3, (d).

6. **Sets, groups, rings, operators.** If A and B are two sets of elements, we may define their (discrete) tensor product as the set of all symbols $\pm a_1 \cdot b_1 \pm \cdots \pm a_n \cdot b_n$, with the obvious definition of $+$, which we assume commutative. This is a free group, generated by all $a \cdot b$; if A and B have m and n elements, respectively, then $A \circ B$ has mn generators.

If G is an Abelian group and A is a set of elements, their tensor product is the set of all $\sum g_i \cdot a_i$, with the distributive law as in (3.1), postulating that $+$ is commutative, and $0 \cdot a + g \cdot a' = g \cdot a'$. This is the "group of all linear forms over elements of A, with coefficients in G". An example is given by the groups of chains used in topology.

We shall say an Abelian group G *admits the ring R as operator ring*, or *admits R* simply, if R has a unit 1, and $rg = g'$ is defined satisfying

$$(6.1) \qquad \begin{aligned} &r(g + g') = rg + rg', \qquad (r + r')g = rg + r'g, \\ &r(r'g) = (rr')g \text{ or } (r'r)g, \qquad 1g = g. \end{aligned}$$

We call R a *left* or *right* operator according as we use $(rr')g$ or $(r'r)g$ in the third relation. In the second case, we might write gr in place of rg, obtaining $(gr')r = g(r'r)$. Suppose, for definiteness, we write $r[g]$ instead of rg. Then a ring can operate on itself in both ways, using

$$(6.2) \qquad r[r'] = rr' \quad \text{and} \quad r[r'] = r'r.$$

The associative law $r[r'[r'']] = (r[r'])[r'']$ holds in either case.

If G and H both admit R, to left or right, we say an isomorphism ϕ between G and H is an *operator isomorphism* if $\phi(rg) = r\phi(g)$; we use $\approx$ again, and say ϕ *preserves* the operator.

THEOREM 12. *If R is a ring with unit, and we define $R \circ G$, considering R as a group under addition, then $R \circ G$ admits R to left or right, under the definitions*

$$(6.3) \qquad r(\sum r_i \cdot g_i) = \sum rr_i \cdot g_i \quad \text{or} \quad \sum r_i r \cdot g_i .$$

The proof is simple. The following theorem is a generalization.

THEOREM 13. *If G admits R to left or to right, then so does any tensor product $G \circ H$ or $H \circ G$, under the definition*

$$(6.4) \qquad r(\sum g_i \cdot h_i) = \sum rg_i \cdot h_i , \qquad r(\sum h_i \cdot g_i) = \sum h_i \cdot rg_i .$$

Suppose G and H both admit R, each to one side. Then we define the *reduced tensor product* $G \circ' H$ with respect to R as follows. Take the tensor product $G \circ H$, and define a new relation

$$rg\cdot h = g\cdot rh. \tag{6.5}$$

$G \circ' H$ is the group thus formed; it is the difference group of $G \circ H$ with the group generated by all $rg\cdot h - g\cdot rh$.

THEOREM 14. *If G admits R to the left, then there is a natural operator isomorphism*

$$R \circ' G \approx G,$$

letting R act on itself to the right and on $R \circ' G$ to the left, given by

$$\phi(\sum r_i\cdot g_i) = \sum r_i g_i . \tag{6.6}$$

Here, (6.5) is replaced by

$$rr'\cdot g = r'[r]\cdot g = r\cdot r'[g] = r\cdot r'g. \tag{6.5'}$$

To show that ϕ is uniquely defined, we have for instance

$$\phi(rr'\cdot g) = (rr')g = r(r'g) = \phi(r\cdot r'g).$$

ϕ is a homomorphism into the whole of G; for $\phi(1\cdot g) = 1g = g$. It preserves the operator, for

$$\phi(r(\sum r_i\cdot g_i)) = \phi(\sum rr_i\cdot g_i) = \sum (rr_i)g_i = \sum r(r_i g_i)$$
$$= r(\sum r_i g_i) = r\phi(\sum r_i\cdot g_i).$$

Finally, ϕ is (1-1). For if $\phi(\sum r_i\cdot g_i) = \sum r_i g_i = 0$, then

$$\sum r_i\cdot g_i = \sum 1\cdot r_i g_i = 1\cdot \sum r_i g_i = 1\cdot 0 = 0.$$

The theorem clearly holds with "right" and "left" interchanged.

Suppose R and S are rings.[10] Then we can make $R \circ S$ a ring in four different ways, namely,

$$(r\cdot s)(r'\cdot s') = rr'\cdot ss' \quad \text{or} \quad rr'\cdot s's, \text{ etc.,}$$
$$(\sum r_i\cdot s_i)(\sum r_j'\cdot s_j') = \sum\sum (r_i\cdot s_i)(r_j'\cdot s_j'). \tag{6.7}$$

The uniqueness of the definition is easily established. The associative and distributive laws hold. If R and S have units 1_R and 1_S, then so has $R \circ S$, namely, $1_R\cdot 1_S$.

We shall not discuss the questions of zero-divisors or of fields.

7. Rational multipliers and tensor products.

Definition. For any rational number r, $r = a/b$, $(a, b) = 1$, and any $A \subset G$

[10] Compare J. L. Dorroh, loc. cit.

(including $A = g$), we let rA be the set of all elements g' such that $bg' = ag$, g in A. This agrees with the definition of aA and with the natural definition of $(1/a)A$. Then some of the formal properties of rational numbers as multipliers hold. In particular, some elements can be divided by certain integers. *Division by integers, when it exists, is unique if and only if G has no elements $\neq 0$ of finite order.* For if g' and g'' are in $(1/a)g$, $g' \neq g''$, then $a(g' - g'') = g - g = 0$, so that $g' - g''$ is of finite order; if $g \neq 0$ is of finite order a, then $(1/a)0$ is not unique. We shall say G has *unique division* if it is completely divisible and has no elements $\neq 0$ of finite order. Because of Theorem 15 below, we may then multiply by rational numbers in such a group, and all formal laws will hold.

The only theorem we will need in §8 is the following.

THEOREM 15. *The following three statements are equivalent*:

(a) *G admits Rt as operator ring; we shall write $r[g]$.*

(b) *G has unique division.*

(c) *For each rational r and each g in G, rg is a unique element of G.*

Further, G can admit Rt in at most one way; if it does, then $rg = r[g]$.

First, if G admits Rt, then G has no elements of finite order. For, note first that (for $a > 0$, and hence for $a \leqq 0$),

$$(*) \qquad a[g] = (1 + \cdots + 1)[g] = 1[g] + \cdots + 1[g] = ag.$$

Now if $ag = 0$, $a \neq 0$, then $a[g] = ag = 0 = a0 = a[0]$; hence

$$g = 1[g] = \left(\frac{1}{a}\,a\right)[g] = \frac{1}{a}[a[g]] = \frac{1}{a}[a[0]] = 1[0] = 0.$$

Next, if (a) holds, then for each integer $a \neq 0$ and each g in G, $g' = (1/a)[g]$ exists, and $ag' = a[g'] = g$; hence (b) holds. (b) clearly implies (c). If (c) holds, then setting $r[g] = rg$ gives (a).

Finally, if two operations $r[g]$ and $r\{g\}$ are defined, then they agree; for by (*),

$$b\left(\frac{a}{b}[g]\right) = \left(b\,\frac{a}{b}\right)[g] = a[g] = ag = b\left(\frac{a}{b}\{g\}\right);$$

as G can have no elements of finite order, $(a/b)[g] = (a/b)\{g\}$. Also

$$b\left(\frac{a}{b}[g]\right) = a[g] = ag = b\left(\frac{a}{b}\,g\right),$$

and hence $r[g] = rg$.

Before considering tensor products, we consider some divisibility properties in general groups. Let δ_r denote the denominator of r; $\delta_r = b$ if $r = a/b$, $(a, b) = 1$.

LEMMA 1. *If rg is not void, then $\delta_r \mid g$, and conversely.*

For if $r = a/b$, $bg' = ag$, and $pa + qb = 1$, then

$$b(qg + pg') = qbg + pag = g.$$

The converse is clear.

LEMMA 2. *If* $(a, b) = 1$, *then*

$$\frac{a}{b} A = a\left(\frac{1}{b} A\right) = \frac{1}{b}(aA). \tag{7.1}$$

To prove the first relation, the elements of $a((1/b)A)$ are all g', $g' = ag^*$, g^* in $(1/b)A$, i.e., $bg^* = g$ in A; then $bg' = ag$, and as $(a, b) = 1$, g' is in $(a/b)A$. Conversely, if g' is in $(a/b)A$, then $bg' = ag$ (g in A). Choose p, q so that $pa + qb = 1$, and set $g^* = qg + pg'$. Then

$$bg^* = qbg + pag = g, \qquad ag^* = qbg' + pag' = g',$$

so that g^* is in $(1/b)A$ and g' is in $ag^* \subset a((1/b)A)$. The second relation is clear.

LEMMA 3. *For any integers a and b,*

$$\frac{1}{a}\left(\frac{1}{b} A\right) = \frac{1}{ab} A, \qquad a\left(\frac{1}{a} A\right) \subset A, \qquad \frac{1}{a}(aA) \supset A. \tag{7.2}$$

The proof is simple.

We turn now to tensor products.

LEMMA 4. *If* $\delta_r \mid g$ *and* $\delta_r \mid h$, *then*

$$g' \cdot h = g \cdot h' \quad \text{for any } g' \text{ in } rg \text{ and any } h' \text{ in } rh. \tag{7.3}$$

Set $r = a/b$, $(a, b) = 1$. If

$$bg' = ag, \qquad g = bg^*, \qquad bh' = ah, \qquad h = bh^*,$$

then

$$g \cdot h' = bg^* \cdot h' = g^* \cdot bh' = g^* \cdot ah = g^* \cdot abh^* = abg^* \cdot h^* = ag \cdot h^*$$
$$= bg' \cdot h^* = g' \cdot bh^* = g' \cdot h.$$

Example. If $\delta_r \mid h$ is false, $rg \cdot h$ may not be uniquely defined. For if $G = H = I_2$, $g = 0_2$, $h = 1_2$, then $G \circ H \approx I_2$, and $\frac{1}{2}g \cdot h$ contains both 0_2 and 1_2.

THEOREM 16. *If* $\delta_r \mid A$ *and* $\delta_r \mid B$, *then*

$$rA \cdot B = A \cdot rB; \tag{7.4}$$

if A and B are single elements, so is $rA \cdot B$.

This follows from Lemmas 1 and 4.

Remark. $r(g \cdot h)$ may be $\neq rg \cdot h$. For example, if $G = H = I_2$, $g = h = 0_2$, $r = \frac{1}{2}$, then $rg \cdot h = 0_2$, while $r(g \cdot h)$ contains both 0_2 and 1_2. However,

$$r(A \cdot B) \supset rA \cdot B; \tag{7.5}$$

for if $r = a/b$, $(a, b) = 1$, g in A, h in B, $bg' = ag$, so that $g' \cdot h$ is in $rA \cdot B$, then

$$b(g' \cdot h) = bg' \cdot h = ag \cdot h = a(g \cdot h) \text{ is in } a(A \cdot B),$$

so that $g' \cdot h$ is in $r(A \cdot B)$.

LEMMA 5. *If $b \mid A$ and $b \mid B$, then*

$$\frac{1}{b}(aA)\cdot B = \frac{a}{b}A\cdot B = a\left(\frac{1}{b}A\right)\cdot B = A\cdot\frac{1}{b}(aB) = A\cdot\frac{a}{b}B = A\cdot a\left(\frac{1}{b}B\right); \tag{7.6}$$

if A and B are single elements, so is the above.

Say $(a, b) = k$, $a = a'k$, $b = b'k$; then $(a', b') = 1$. To prove the first relation, we use Lemmas 2 and 3 and Theorem 16, and the fact $b \mid aA$:

$$\frac{1}{b}(aA)\cdot B = \frac{1}{b'}\left(\frac{1}{k}(k(a'A))\right)\cdot B \supset \frac{1}{b'}(a'A)\cdot B = \frac{a'}{b'}A\cdot B = \frac{a}{b}A\cdot B,$$

$$\frac{1}{b}(aA)\cdot B = A\cdot a\left(\frac{1}{b}B\right) = A\cdot a'\left(k\left(\frac{1}{k}\left(\frac{1}{b'}B\right)\right)\right) \subset A\cdot a'\left(\frac{1}{b'}B\right) = \frac{a}{b}A\cdot B.$$

From these the relation follows. The other relations are consequences of this one or are easily proved. The last statement follows from Theorem 16.

THEOREM 17. *If $\delta_r\delta_{r'} \mid A$ and $\delta_r\delta_{r'} \mid B$,[11] then*

$$r(r'A)\cdot B = (rr')A\cdot B = A\cdot(rr')B, \text{ etc.}; \tag{7.7}$$

if A and B are single elements, so is the above.

Say $r = a/b$, $r' = c/d$, $(a, b) = (c, d) = 1$. As $bd \mid cA$, etc.,

$$r(r'A)\cdot B = a\left(\frac{1}{b}\left(\frac{1}{d}(cA)\right)\right)\cdot B = \frac{1}{bd}(cA)\cdot aB = ac\left(\frac{1}{bd}A\right)\cdot B$$

$$= \frac{ac}{bd}A\cdot B = (rr')A\cdot B, \text{ etc.}$$

8. **The tensor product** $Rt \circ G$. First note that, if F is any completely divisible group (in particular, Rt), then in studying $F \circ G$, we could assume that G has no elements $\neq 0$ of finite order. For otherwise, let G' be the subgroup of elements of finite order of G. As G' is in the nullifier of F, $\sum^*(F\cdot G') = 0$ (see Theorem 9); hence, by Theorem 10,

$$F \circ G \approx F \circ (G \ominus G').$$

Thus we may replace G by $G \ominus G'$, which has no elements $\neq 0$ of finite order.

THEOREM 18. *In $Rt \circ G$, each element may be written in the form $(1/a)\cdot g$. If G has no elements $\neq 0$ of finite order, then $r\cdot g = 0$ if and only if $r = 0$ or $g = 0$.*

First,

$$\sum r_i\cdot g_i = \sum \frac{a_i}{a}\cdot g_i = \frac{1}{a}\cdot\sum a_i g_i = \frac{1}{a}\cdot g.$$

Next, suppose we have an equivalence sequence reducing $r\cdot g$ to $0\cdot 0$. In all terms occurring, there is a least common denominator c. Multiplying every-

[11] Possibly this hypothesis can be weakened.

thing by c gives an equivalence sequence, which may be interpreted as a sequence in $I_0 \circ G$, or again, in G itself. Hence, if $r = a/b$, we have $(ca/b)g = 0$. If $r \neq 0$, then $ca/b \neq 0$, and as G has no elements of finite order, $g = 0$.

THEOREM 19. *$Rt \circ G$ has unique division.*

This follows from Theorems 12 and 15.

THEOREM 20. *There is an isomorphism $G \approx Rt \circ G$, given by $\phi(\sum r_i \cdot g_i) = \sum r_i g_i$, if and only if G has unique division.*

This is an extension of Theorem 15. One half follows from Theorem 19; the other half is clear.

THEOREM 21. *If G has no elements $\neq 0$ of finite order, then $Rt \circ G$ is the smallest completely divisible group containing G. That is, if H is completely divisible and contains a subgroup $H_1 \approx G$, then it contains a subgroup $H_2 \approx Rt \circ G$.*

Let H' be the subgroup of elements of finite order of H. Clearly H' is completely divisible; hence we may write $H = H' \oplus H''$.[12] For any $h = h' + h''$, write $h' = \phi(h)$, $h'' = \psi(h)$; then ϕ and ψ are homomorphisms. Set $H_1'' = \psi(H_1)$; then $H_1'' \approx G$. For if $\psi(h_1) = 0$ (h_1 in H_1), then h_1 is in H', and hence is of finite order; but h_1 is in $H_1 \approx G$, which gives $h_1 = 0$.

Let H_2 be the subgroup of H'' containing all elements with multiples in H_1''. H_2 is completely divisible. For given h in H_2 and an integer $a \neq 0$, choose h^* in H so that $ah^* = h$, and set $h_1 = \psi(h^*)$. Then h_1 is in H'', and as h is in H'',

$$ah_1 = a\psi(h^*) = \psi(ah^*) = \psi(h) = h;$$

hence h_1 is in H_2. As H'' has no elements $\neq 0$ of finite order, neither has H_2; hence H_2 has unique division.

Let θ be the isomorphism of G into H_1''. As rh is uniquely defined for h in the group H_2 (Theorem 15), and clearly obeys $(r + r')/h = rh + r'h$, $r(h + h') = rh + rh'$, we may set

$$\Theta(\sum r_i \cdot g_i) = \sum r_i \theta(g_i),$$

defining a homomorphism of $Rt \circ G$ into H_2. Suppose $\Theta(\alpha) = 0$. If $\alpha = (1/a) \cdot g$ (Theorem 18), then $\Theta(\alpha) = (1/a)\theta(g) = 0$. Multiplying by a gives $\theta(g) = 0$, and hence $g = 0$, and $\alpha = 0$, as θ is an isomorphism. Hence Θ is (1-1). For any h in H_2, we may take a so that ah is in H_1''; then for some g, $ah = \theta(g) = \Theta(1 \cdot g)$, and $h = \Theta((1/a) \cdot g)$; hence Θ is an isomorphism, and the theorem is proved.

9. **Tensor products and character groups.** In some cases, the group $Ch_H(G)$ of homomorphisms of G into H can be expressed in terms of the two groups H and $Ch_{I_0}(G)$, by (9.1). See also Theorem 25 of Part II. We remark in passing that $Ch_H(G)$ and G form a group pair with respect to H, with the definition $\Phi(\sum \phi_i \cdot g_i) = \sum \phi_i(g_i)$ (ϕ_i in $Ch_H(G)$, g_i in G).

[12] See R. Baer, *The subgroup of elements of finite order of an Abelian group*, Annals of Mathematics, vol. 37(1936), pp. 766–781, (1; 1).

THEOREM 22.[13] *There is a natural isomorphism*

$$Ch_{I_0}(G) \circ H \approx Z \subset Ch_H(G), \tag{9.1}$$

defined as follows. For u_i in $Ch_{I_0}(G)$ and h_i in H,

$$\Phi(\sum u_i \cdot h_i ; g) = \sum u_i(g) h_i . \tag{9.2}$$

If either G or H is a free group with a finite number of generators, then $Z = Ch_H(G)$.

It is clear that the definition of Φ is unique, and Φ is a homomorphism. We must show that it is (1-1). Suppose the element (9.2) equals 0. Say the sum contains n terms. Let $A = I_0 \oplus \cdots \oplus I_0$ be the group of all n-tuples $(a_1, \cdots, a_n)$ of integers, and let A' be the subgroup of all $(a_1, \cdots, a_n)$ in A for which $\sum a_i h_i = 0$. We may choose a base

$$\alpha_1, \cdots, \alpha_n ; \qquad \alpha_i = (a_{i1}, \cdots, a_{in})$$

in A and integers $p_1, \cdots, p_m$ $(m \leqq n)$ such that

$$p_1 \alpha_1, \cdots, p_m \alpha_m$$

form a base in A'.[14] For each g, let $u(g)$ be the element $(u_1(g), \cdots, u_n(g))$ of A; as $\sum u_i(g) h_i = 0$, $u(g)$ is in A'. Hence, for each g, there is a uniquely defined set of numbers $\rho_1(g), \cdots, \rho_m(g)$ such that

$$u(g) = \sum_{j=1}^{m} \rho_j(g) p_j \alpha_j ;$$

hence

$$u_i = \sum_{j=1}^{m} p_j a_{ji} \rho_j .$$

As the $u_i(g)$ are homomorphisms, so are $u(g)$ and the $\rho_i(g)$; the $\rho_i(g)$ are in $Ch_{I_0}(G)$. Set

$$\bar{h}_i = \sum_{k=1}^{n} a_{ik} h_k \qquad (i = 1, \cdots, m);$$

then

$$p_i \bar{h}_i = \sum_{k=1}^{n} p_i a_{ik} h_k = 0 \qquad (i = 1, \cdots, m),$$

by the choice of the α_i and p_i. Hence, using the distributive laws in $Ch_{I_0}(G) \circ H$,

$$\sum_{i=1}^{n} u_i \cdot h_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} p_j a_{ji} \rho_j \right) \cdot h_i = \sum_{j=1}^{m} \left(\rho_j \cdot p_j \sum_{i=1}^{n} a_{ji} h_i \right)$$
$$= \sum_{j=1}^{m} (\rho_j \cdot p_j \bar{h}_j) = \sum_{j=1}^{m} (\rho_j \cdot 0) = 0,$$

as required.

[13] Compare Theorem 25.

[14] See, for example, Alexandroff-Hopf, loc. cit., p. 566.

Now suppose H has a base $\bar{h}_1, \cdots, \bar{h}_n$, so any h may be written uniquely $\sum a_i\bar{h}_i$. Let ϕ be any homomorphism of G into H; then we may write

$$\phi(g) = \sum u_i(g)\bar{h}_i,$$

and the $u_i(g)$ are elements of $Ch_{I_0}(G)$. Also

$$\Phi(\sum u_i\cdot\bar{h}_i; g) = \sum u_i(g)\bar{h}_i = \phi(g),$$

so Φ maps $Ch_{I_0}(G) \circ H$ into the whole of $Ch_H(G)$.

Suppose finally that G has a base $\bar{g}_1, \cdots, \bar{g}_n$. Let $\bar{u}_i(g)$ be the element of $Ch_{I_0}(G)$ defined by $\bar{u}_i(\bar{g}_i) = 1$, $\bar{u}_i(\bar{g}_j) = 0$ $(j \neq i)$. Take any homomorphism ϕ of G into H. Then for any $g = \sum a_i\bar{g}_i$, $\bar{u}_i(g) = a_i$, and

$$\phi(g) = \sum a_i\phi(\bar{g}_i) = \sum \bar{u}_i(g)\phi(\bar{g}_i);$$

hence, setting $h_i = \phi(\bar{g}_i)$,

$$\Phi(\sum \bar{u}_i\cdot h_i; g) = \sum \bar{u}_i(g)\phi(\bar{g}_i) = \phi(g).$$

This completes the proof.

Examples. Suppose $G = H = I_2$. Then $Ch_H(G)$ has two elements, while $Ch_{I_0}(G) \circ H$ has only one. Again, let G be the additive group of triadic rational numbers (all numbers of the form $a/3^b$), and set $H = I_2$. There are two elements in $Ch_H(G)$, determined by $\phi(1) = 0_2$ and $\phi(1) = 1_2$; but there is only one element in $Ch_{I_0}(G) \circ H$.

II. Linear spaces

10. Products, finite dimensional spaces. A *linear space*, or *vector space*, G, is an Abelian[15] group which admits the real numbers Rl as operators (see §6). Let $G(g_1, \cdots, g_m)$ be the subspace of G generated by $g_1, \cdots, g_m$, i.e., all $\sum a_ig_i$ (a_i real). If such a set generates G itself, then let $g_1, \cdots, g_m$ be such a set with the least number of elements. Then these elements form a *base* for G, and G is of *dimension* m.

In any finite dimensional linear space G, with a base $g_1, \cdots, g_m$, we may introduce a *natural topology* by defining neighborhoods $U(\epsilon)$ of 0 for each $\epsilon > 0$, consisting of all $\sum a_ig_i$ with $\sum a_i^2 < \epsilon^2$. The topology is independent of the choice of a base. In this topology, the operation ag is continuous in both variables.

In the tensor product $G \circ H$, we clearly wish to have

$$a(g\cdot h) = ag\cdot h = g\cdot ah \qquad (a \text{ in } Rl); \tag{10.1}$$

hence we use the *reduced tensor product* (see (6.4)), but call it the *tensor product* simply. Without this, we would have for instance in Rl, $\sqrt{2}\cdot 1 \neq 1\cdot\sqrt{2}$. Further, *if we assume that $g\cdot h$ is continuous, then* (10.1) *follows.* To show this, the last statement in Theorem 15, and Theorem 16, show that $bg\cdot h = g\cdot bh$ for any rational b. Letting $b \to a$ gives the result.

[15] The group is necessarily Abelian. Compare §3, (e). If G is not linear, it can be made so by taking $Rl \circ G$; see Theorem 12, §6.

We assume in the rest of §10 that G and H have bases $\bar{g}_1, \cdots, \bar{g}_m$ and $\bar{h}_1, \cdots, \bar{h}_n$, respectively.

THEOREM 23. *An element of* $G \circ H$ *may be written uniquely in any one of the three normal forms*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(\bar{g}_i \cdot \bar{h}_j) = \sum_{i=1}^{m} \bar{g}_i \cdot h_i' = \sum_{j=1}^{n} g_j' \cdot \bar{h}_j. \tag{10.2}$$

For, if

$$g_k = \sum b_{ki}\bar{g}_i, \qquad h_k = \sum c_{ki}\bar{h}_i, \tag{10.3}$$

then the distributive laws give

$$\sum_k g_k \cdot h_k = \sum_k \left(\sum_i b_{ki}\bar{g}_i\right) \cdot \left(\sum_j c_{kj}\bar{h}_j\right) = \sum_k \sum_i \left(\bar{g}_i \cdot \sum_j b_{ki}c_{kj}\bar{h}_j\right)$$
$$= \sum_i \bar{g}_i \cdot \sum_j \sum_k b_{ki}c_{kj}\bar{h}_j, \text{ etc.};$$

thus (10.2) holds with

$$a_{ij} = \sum_k b_{ki}c_{kj}, \qquad h_i' = \sum_j a_{ij}\bar{h}_j, \qquad g_i' = \sum_j a_{ji}\bar{g}_j. \tag{10.4}$$

Given any expression $\sum g_i \cdot h_i$ for α in $G \circ H$, the above procedure gives the normal forms in a unique manner; we must show that if $\sum g_i \cdot h_i = \sum g_i^* \cdot h_i^*$, the two expressions give the same result. It is sufficient to prove this for $(g + g^*) \cdot h$ and $g \cdot h + g^* \cdot h$, for $g \cdot (h + h^*)$ and $g \cdot h + g \cdot h^*$, and for $ag \cdot h$ and $g \cdot ah$. In each case, the proof is simple.

Let $Ch_H(G)$ denote the group of linear maps (= continuous homomorphisms) of G into H; this is a linear space of dimension mn. In particular, $L(G) = Ch_{Rl}(G)$ is the group of linear real-valued functions in G, and is called the *conjugate space* of G. Here, isomorphism will mean continuous isomorphism = operator isomorphism. The following theorem is well known.

THEOREM 24. $L(G) \approx G$. *Further, there is a natural isomorphism*

$$L(L(G)) \approx G, \tag{10.5}$$

defined as follows. For any g in G, $\phi(g)$ is the element of $L(L(G))$ which, for any u in $L(G)$, has the value $u(g)$.

Let $\bar{u}_i(g)$ be the element of $L(G)$ such that $\bar{u}_i(\bar{g}_i) = 1$, $\bar{u}_i(\bar{g}_j) = 0$ $(j \neq i)$. Clearly $\bar{u}_1, \cdots, \bar{u}_m$ form a base in $L(G)$; hence $L(G) \approx G$. Next, ϕ is linear. It is (1-1); for if $\phi(g) = 0$, then $u(g) = 0$ (all u in $L(G)$), which implies $g = 0$. Given any v in $L(L(G))$, set $a_i = v(\bar{u}_i)$; then for any $u = \sum b_i\bar{u}_i$,

$$u\left(\sum a_i\bar{g}_i\right) = \sum_i a_i \sum_j b_j\bar{u}_j(\bar{g}_i) = \sum a_ib_i = \sum b_iv(\bar{u}_i) = v(u),$$

so that $\phi(\sum a_i\bar{g}_i) = v$. Clearly $\phi(ag) = a\phi(g)$; hence ϕ is an isomorphism.

THEOREM 25.[16] *There is a natural isomorphism*

$$Ch_H(G) \approx L(G) \circ H, \tag{10.6}$$

given by

$$\phi(\sum u_i \cdot h_i \,;\, g) = \sum u_i(g) h_i . \tag{10.7}$$

ϕ is clearly uniquely defined. If we write all elements of $L(G) \circ H$ in the third normal form $\sum u_i \cdot \bar{h}_i$, the properties of ϕ are easily established; for any element of $Ch_H(G)$ can be written uniquely as $\sum u_i(g)\bar{h}_i$, and if this is the zero element, i.e., it is equal to zero in H for all g, then all $u_i(g) = 0$.

COROLLARY I. *$G \circ H$ may be written in the form*

$$G \circ H \approx L(L(G)) \circ H \approx Ch_H(L(G)). \tag{10.8}$$

The isomorphism of the first group into the last is given as follows. For $\sum g_i \cdot h_i$ in $G \circ H$ and u in $L(G)$,

$$\phi(\sum g_i \cdot h_i \,;\, u) = \sum u(g_i) h_i . \tag{10.9}$$

COROLLARY II. *There is a natural isomorphism*

$$Ch_G(Rl) \approx G; \tag{10.10}$$

for u in $Ch_G(Rl)$, $\phi(u) = u(1)$.

For $L(Rl) \circ G \approx Rl \circ G \approx G$. (Moreover, a direct proof is obvious.)

THEOREM 26. *$G \circ H$ is a linear space of dimension mn, with a base $\bar{g}_1 \cdot \bar{h}_1, \cdots, \bar{g}_m \cdot \bar{h}_n$. If $\{U\}$ and $\{V\}$ are neighborhood systems in G and H, respectively, defining their natural topologies, then either of the following neighborhood systems, if we use $p = \min(m, n)$,*

$$N(U, V) = U \cdot V + \cdots + U \cdot V \qquad (p \text{ summands}), \tag{10.11}$$

$$N(U_1, U_2, \cdots ; V_1, V_2, \cdots) = \sum_k{}^*(U_k \cdot V_k) \tag{10.12}$$

defines the natural topology in $G \circ H$.[17] *The multiplication $g \cdot h$ is continuous.*

The first part of the theorem follows from Theorem 23. Let N, N', N'' denote natural neighborhoods and those of (10.11) and (10.12). Given an $N = N(\epsilon)$, consisting of all $\sum a_{ij}\bar{g}_i \cdot \bar{h}_j$ with $\sum a_{ij}^2 \leqq \epsilon^2$, set $\epsilon_1 = \epsilon/(mn)^{\frac{1}{2}}$, and let

$$U_k = U(\epsilon_1/2^k), \qquad V_k = V(1), \qquad (k = 1, 2, \cdots),$$

be natural neighborhoods in G and H. Then if $g_k = \sum b_{ki}\bar{g}_i$ is in U_k and $h_k = \sum c_{kj}\bar{h}_j$ is in V_k, (10.4) gives, if we use any finite number ν of summands in (10.12),

$$|a_{ij}| = \left| \sum_k b_{ki} c_{kj} \right| < \sum_{k=1}^{\nu} \epsilon_1/2^k < \epsilon_1 = \epsilon/(mn)^{\frac{1}{2}}.$$

[16] This holds if at least one of G, H is of finite dimension. Compare Theorem 22.

[17] If we map Rl into a curve everywhere dense on the torus, the topology of the torus gives an "unnatural" topology in Rl. In $Rl \circ Rl$, either type of neighborhood as here given then contains the whole space.

Hence $\sum a_{ij}^2 < \epsilon^2$, and $\sum_{k=1}^{\nu} g_k \cdot h_k$ is in N. Thus any N contains an N''.

Next, given an N'', take

$$U \subset U_1 \cap \cdots \cap U_p, \qquad V \subset V_1 \cap \cdots \cap V_p.$$

Then clearly $N' = N(U, V) \subset N''$.

Next, take any $N' = N(U, V)$. Suppose for definiteness that $p = m$. Take ϵ_1 so that $U(2\epsilon_1) \subset U$ and $V(\epsilon_1) \subset V$, and set $\epsilon = \epsilon_1^2$. Now take any α of $G \circ H$ in $N(\epsilon)$; then we can write $\alpha = \sum a_{ij}\bar{g}_i \cdot \bar{h}_j$, with $\sum a_{ij}^2 < \epsilon^2$. Also,

$$\alpha = \sum_{i=1}^{m} \left(\epsilon_1 \bar{g}_i \cdot \sum_{j=1}^{n} \theta a_{ij} \bar{h}_j \right), \qquad \theta = \frac{1}{\epsilon_1}.$$

As $m = p$ and $\epsilon_1 \bar{g}_i$ is in $U(2\epsilon_1) \subset U$, to show that $N(\epsilon) \subset N(U, V)$, it is sufficient to show that $\sum_j \theta a_{ij} \bar{h}_j$ is in $V(\epsilon_1)$. But

$$\sum_j \theta^2 a_{ij}^2 \leqq \theta^2 \sum_{i,j} a_{ij}^2 < \theta^2 \epsilon^2 = \epsilon_1^2,$$

and this proves the statement.

The continuity of $g \cdot h$ is clear from the relation

$$\sum (a_i + a_i')\bar{g}_i \cdot \sum (b_j + b_j')\bar{h}_j - \sum a_i \bar{g}_i \cdot \sum b_j \bar{h}_j = \sum (a_i' b_j + a_i b_j' + a_i' b_j')\bar{g}_i \cdot \bar{h}_j.$$

If G and H are metric, and hence scalar products $g \circ g'$ and $h \circ h'$ are defined, we may define scalar products and hence a metric in $G \circ H$ by

$$(10.13) \qquad \left(\sum_k g_k \cdot h_k\right) \circ \left(\sum_l g_l' \cdot h_l'\right) = \sum_{k,l} (g_k \circ g_l')(h_k \circ h_l').$$ [18]

11. **Tensor algebra.** Let G be a linear space of finite dimension n; in §12, it will be the "tangent space" at a point of a manifold. Any element of G we shall call a *contravariant vector*. An element of $H = L(G)$ we call a *covariant vector*. Any element of the linear space

$$(11.1) \qquad T(p, q) = G \circ \cdots \circ G \circ H \circ \cdots \circ H \qquad (p \text{ factors } G,\ q \text{ factors } H)$$

we shall call a *tensor of contravariant order p and covariant order q*. As $L(p, q)$ is a linear space, we may *add* two tensors of the same type, and *multiply* a tensor by a real number. Using Theorems 3 and 4 in Part I, we have

$$(G \circ \cdots \circ H \circ \cdots) \circ (G \circ \cdots \circ H \circ \cdots) \approx G \circ \cdots \circ G \circ \cdots \circ H \circ \cdots \circ H \circ \cdots.$$

Hence a tensor of $T(p, q)$ and a tensor of $T(p', q')$ may be *multiplied*, giving a tensor of $T(p + p', q + q')$.

The process of *contraction* is as follows. To contract the element $g \cdot h$ of

[18] For a study of this metric in Hilbert spaces, see Murray and von Neumann, loc. cit.

$T(1, 1) = G \circ H$, recall that $H = L(G)$, and set $\phi(g \cdot h) = h(g)$, a real number. To contract the element

$$\alpha = \sum_k g_k^1 \cdots g_k^p \cdot h_k^1 \cdots h_k^q \quad \text{of } T(p, q)$$

with respect to the p-th g and the q-th h, for example, set

$$\phi(\alpha) = \sum_k h_k^q(g_k^p) g_k^1 \cdots g_k^{p-1} \cdot h_k^1 \cdots h_k^{q-1}; \tag{11.2}$$

this is an element of $T(p - 1, q - 1)$.

Let $\bar{g}_1, \cdots, \bar{g}_n$ form a base in G, and choose $\bar{h}^i$ so that $\bar{h}^i(\bar{g}_j) = \delta_j^i$; then $\bar{h}^1, \cdots, \bar{h}^n$ form a base in H. By the proof of Theorem 23, we may write any element of $T(p, q)$ uniquely in the normal form

$$\alpha = \sum_{i_r, j_s = 1}^{n} A_{j_1 \cdots j_q}^{i_1 \cdots i_p} \bar{g}_{i_1} \cdots \bar{g}_{i_p} \bar{h}^{j_1} \cdots \bar{h}^{j_q}; \tag{11.3}$$

there are n^{p+q} terms in the sum, and the $A_{j_1 \cdots j_q}^{i_1 \cdots i_p}$ are called the *components of* α in the coördinate system of the $\bar{g}_i$. Let us verify the *laws of transformation* of the components. Suppose we introduce the new base $g_1', \cdots, g_n'$. Say

$$\bar{g}_i = \sum_{k=1}^{n} a_i^k g_k', \qquad g_i' = \sum_{k=1}^{n} a_i'^k \bar{g}_k .$$

If $h'^i(g_j') = \delta_j^i$, then setting $h'^i = \sum b_k^i \bar{h}^k$ gives

$$\delta_j^i = h'^i(g_j') = \sum_k b_k^i \bar{h}^k(\sum_l a_j'^l \bar{g}_l) = \sum_{k,l} b_k^i a_j'^l \delta_l^k = \sum_k b_k^i a_j'^k.$$

Hence $b_j^i = a_j^i$, and $\bar{h}^i = \sum a_k'^i h'^k$. Putting in (11.3) and using the distributive laws gives

$$\alpha = \sum A_{j_1 \cdots j_q}^{i_1 \cdots i_p} a_{i_1}^{k_1} \cdots a_{i_p}^{k_p} a_{l_1}'^{j_1} \cdots a_{l_q}'^{j_q} g_{k_1}' \cdots g_{k_p}' \cdot h'^{l_1} \cdots h'^{l_q}.$$

Calling the new components $A_{l_1 \cdots l_q}'^{k_1 \cdots k_p}$, we have the ordinary laws of transformation. Note that

$$h(g) = \sum_i B_i \bar{h}^i(\sum_j A^j \bar{g}_j) = \sum_{i,j} A^j B_i \bar{h}^i(\bar{g}_j) = \sum_i A^i B_i,$$

so that the terms as here introduced agree with the usage in tensor algebra.

12. **Tensor analysis.** Let M be a differentiable manifold.[19] By a *parametrized curve* C starting at the point x_0 in M we shall mean a differentiable map ϕ of an interval $0 \leqq t \leqq \eta$ into M, with $\phi(0) = x_0$. Let us introduce a coördinate system into a neighborhood of M about x_0, i.e., a (1-1) differentiable map θ of a region of the space E of sets of n numbers $(x^1, \cdots, x^n)$ into M, with non-vanishing Jacobian; say $\theta(0, \cdots, 0) = x_0$. Then C translates into a curve C' in E,

[19] See, for instance, O. Veblen-J. H. C. Whitehead, *Foundations of Differential Geometry*, Cambridge Tracts in Mathematics, No. 29, 1933, or H. Whitney, *Differentiable manifolds*, Annals of Mathematics, vol. 37(1936), pp. 645–680.

given by $\theta^{-1}(\phi(t))$, if η is small enough. We say two parametrized curves starting at x_0 are *equivalent* if, when translated into E, they have the same tangent vector (in both magnitude and direction). Clearly the definition of equivalence is independent of the coördinate system chosen. Hence the classes of equivalent curves form a set of elements intrinsically defined in M; we call these *contravariant vectors* at x_0. Using a fixed coördinate system, we may obtain a (1-1) correspondence between contravariant vectors g at x_0 and vectors v in E at 0, merely by choosing, as an interval, the line segment of v, parametrized so that $t = 1$ at its end, and mapping it (or a portion of it, if it does not lie wholly in the region) into M with θ. We may *add* two contravariant vectors at x by taking the corresponding vectors in E, adding, and mapping back into M. Again the result is independent of the coördinate system chosen; hence *the contravariant vectors at x_0 form an intrinsically defined linear space,* the *tangent space $G(x_0)$ to M at x_0*.

We may obtain an intrinsic definition of $L(G(x_0)) = H(x_0)$ at x_0 by considering differentiable functions defined in a neighborhood of x_0, which vanish at x_0, and calling two functions equivalent if their partial derivatives at x_0 are the same in any coördinate system. To add covariant vectors, we need merely add the corresponding functions.

We shall consider briefly *covariant differentiation* in M. Suppose that to any two sufficiently near points x_0 and x_1 of M corresponds a linear map $\Psi_{x_1 x_0}$ of $G(x_1)$ into $G(x_0)$, so that certain simple continuity and linearity properties are satisfied, which we shall not make precise. This will define an *affine connection*[20] in M. Now let $A(x)$ be a differentiable tensor field, being, for each x, an element of $T(p, q; x)$ (using $G(x)$). Let g be any contravariant vector at x_0, and let C, given by $\phi(t)$, be a corresponding parametrized curve. Then if $x_t = \phi(t)$, we may define

$$(12.1) \qquad \nabla_g A(x_0) = \lim_{t \to 0} \frac{1}{t} [\Psi_{x_t x_0} A(x_t) - A(x_0)].$$

(Of course $\Psi_{x_t x_0}$ may be used to translate a tensor at x_t into a tensor at x_0.) For each g at x_0, $\nabla_g A(x_0)$ is a tensor of $T(p, q; x_0)$, and it depends linearly on g; hence we have a linear map of $G(x_0)$ into $T(p, q; x_0)$. By Theorem 25, there is a natural isomorphism

$$Ch_{T(p,q,x_0)}(G(x_0)) \approx T(p, q; x_0) \circ L(G(x_0)) \approx T(p, q + 1; x_0).$$

Hence, at each point x_0 we have a tensor of $T(p, q + 1; x_0)$, of the same contravariant order as A and of covariant order one greater; this is the *covariant derivative* of A at x_0. Again, the definition is intrinsic.

[20] By using a coördinate system about x_0 and letting $x_1 \to x_0$, we may use this connection to obtain an affine connection in the ordinary sense. Conversely, given an ordinary affine connection, we may define geodesics in M, and by following along them, define a connection as above. If we imbed M in a Euclidean space as in Whitney, loc. cit., Theorem 1, we may realize the tangent spaces by tangent planes of dimension n, and define an affine connection by projecting one tangent plane onto another.

13. **Products, general linear spaces.** A representation $\sum g_i \cdot h_i$ of an element α of $G \circ H$ is *minimal* if there is no representation with fewer summands. The *rank* $\rho(\alpha)$ of α is the number of summands in a minimal representation of α. We consider $0 = 0 \cdot 0$ as having no summands, and set $\rho(0) = 0$.

We collect some known results (at least for finite dimensional spaces) in the following theorem.

THEOREM 27. *$G \circ H$ is a linear space. For any α in $G \circ H$ there are corresponding linear subspaces $G(\alpha)$ and $H(\alpha)$ of G and H with the following properties.*

(a) *There is a representation $\sum g_i \cdot h_i$ for α with g_i in $G(\alpha)$, h_i in $H(\alpha)$. In any representation $\sum g_i' \cdot h_i'$ for α, $G(\alpha) \subset G(g_1', \cdots)$, $H(\alpha) \subset H(h_1', \cdots)$.*

(b) $\dim G(\alpha) = \dim H(\alpha) = \rho(\alpha)$; *$G(\alpha)$ and $H(\alpha)$ are $G(g_1, \cdots)$ and $H(h_1, \cdots)$ in any minimal representation $\sum g_i \cdot h_i$ of α.*

(c) *$\sum g_i \cdot h_i$ is minimal if and only if the sets $g_1, \cdots$ and $h_1, \cdots$ are each independent.*

(d) *If $g_1, \cdots, g_m$ and $h_1, \cdots, h_n$ are bases in subspaces G' of G and H' of H, and $\alpha = \sum a_{ij} g_i \cdot h_j$, then $\rho(\alpha) = \text{rank} \| a_{ij} \|$.*

(e) *If $g \cdot h = 0$, then either $g = 0$ or $h = 0$.*

(f) *$g \cdot h = g' \cdot h' \neq 0$ if and only if $g' = ag$, $h' = (1/a)g$ for some real a.*

The first statement follows from Theorems 1 and 13.

Suppose $\alpha = \sum g_i' \cdot h_i' = \sum g_i'' \cdot h_i''$, g_i' in G', g_i'' in G'', h_i' and h_i'' in H^*. Set $G^* = G' \cap G''$, and choose subspaces G_1 and G_2 (possibly containing 0 alone) such that

$$G' + G'' = G^* \oplus G_1 \oplus G_2, \qquad G_1 \subset G', \qquad G_2 \subset G''.$$

Choose bases $\{g_i^*\}$ in G^*, $\{g_i^1\}$ in G_1, $\{g_i^2\}$ in G_2; then all the g's form a base in $G' + G''$. By Theorem 23, we may write uniquely, for some h_i^*, etc., in H^*,

$$\alpha = \sum g_i^* \cdot h_i^* + \sum g_i^1 \cdot h_i^1 + \sum g_i^2 \cdot h_i^2.$$

Now $G' = G^* \oplus G_1$; hence, if we reduce $\sum g_i' \cdot h_i'$ to this normal form, the third group of terms will not appear. As the normal form is unique, the third sum $= 0$. Similarly, as $G'' = G^* \oplus G_2$, the second sum vanishes. Hence $\alpha = \sum g_i^* \cdot h_i^*$ can be expressed by using g's from $G' \cap G''$ alone. Hence there is a minimal subspace $G(\alpha)$ which may be used. Find similarly a minimal $H(\alpha)$. Now α can be expressed, by using $G(\alpha)$ and $H' \supset H(\alpha)$, and $G' \supset G(\alpha)$ and $H(\alpha)$. Choosing bases properly in G' and H' and using the first normal form, we see at once that α may be expressed, using $G(\alpha)$ and $H(\alpha)$. This proves (a).

Next we show that rank $\| a_{ij} \|$ depends on α alone. Suppose $\{g_i\}$ and $\{g_i'\}$ are bases in G', $\{h_i\}$ is a base in H', and $\alpha = \sum a_{ij} g_i \cdot h_j = \sum a_{kj}' g_k' \cdot h_j$. If $g_i = \sum_k b_{ki} g_k'$, then $a_{kj}' = \sum_i b_{ki} a_{ij}$, i.e., $A' = BA$. As B is non-singular, rank $A =$ rank A'. Similarly, a change of base in H' causes no change in the rank. If $G'' \supset G'$ and $H'' \supset H'$, and we choose bases in these spaces containing the above g_i and h_i, then $\sum a_{ij} g_i \cdot h_j$ is also a normal form for α, using G'' and H''. The new $\| a_{ij} \|$ is the old $\| a_{ij} \|$ with extra rows and columns of zeros; the ranks are therefore the same. Now given any two representations of α in normal form, using the pair G', H' and the pair G'', H'', we may also write α in normal form, using $G' + G''$ and $H' + H''$. The above proof shows that all ranks of matrices are the same.

If $\sum_{i=1}^{\rho(\alpha)} g_i \cdot h_i$ is minimal, then obviously the sets $\{g_i\}$ and $\{h_i\}$ are independent. They form bases in spaces G' and H', say, and the expression $\sum g_i \cdot h_i$ is then in normal form. The matrix is the unit matrix, and hence is of rank $\rho(\alpha)$. This proves (d). As $G(\alpha) \subset G'$, and dim $G(\alpha) < \rho(\alpha)$ is clearly impossible, $G(\alpha) = G'$ and dim $G(\alpha) = \rho(\alpha)$; similarly for $H(\alpha)$. (b) is now proved. If $\alpha = \sum_{i=1}^{r} g_i \cdot h_i$ and the sets $\{g_i\}$, $\{h_i\}$ are independent, then we have a representation in normal form, with matrix of rank r; hence $r = \rho(\alpha)$, and $\sum g_i \cdot h_i$ is minimal. This proves (c).

To prove (e), suppose $g \neq 0$, $h \neq 0$. Then $g \cdot h$ is minimal, by (c), hence $\rho(g \cdot h) = 1$, and $g \cdot h \neq 0$. (f) follows from the fact that for $\alpha = g \cdot h = g' \cdot h' \neq 0$, $G(\alpha)$ = all multiples of g = all multiples of g'.

THEOREM 28. *If G' is a linear subspace of G, then there is a natural isomorphism* $G' \circ H \approx \sum^*(G' \cdot H)$.

Using $\sum g_i \times h_i$ in $G' \circ H$, set $\phi(\sum g_i \times h_i) = \sum g_i \cdot h_i$. Clearly ϕ is a uniquely defined homomorphism onto the whole of $\sum^*(G' \cdot H)$. Suppose $\phi(\sum g_i \times h_i) = \sum g_i \cdot h_i$, $\sum g_i \times h_i \neq 0$. We may suppose $\sum g_i \times h_i$ is minimal. Then the sets $\{g_i\}$ and $\{h_i\}$ are independent, and hence $\sum g_i \cdot h_i$ is minimal, by the last theorem, and $\sum g_i \cdot h_i \neq 0$. Hence ϕ is (1-1), and this completes the proof.

14. **On topological linear spaces.** We shall use the following definition. If $G' \subset G^*$, a *projection* of G^* into G' is a linear map of G^* into G' such that every element of G' is fixed.

Definition. We shall call a *topological linear space* G a linear space with sets $U, V, \cdots$, called neighborhoods (of 0), such that:

(1) 0 is in every U;

(2) given U, V, there is a $W \subset U \cap V$;

(3) given U, there is a V such that for $-1 \leqq a \leqq 1$, $aV \subset U$;

(4) given U, there is a V with $V + V \subset U$;

(5) for every U and every g in G there is an a with g in aU;

(6) for every finite dimensional subspace G' of G and every natural neighborhood U' in the space G' (see §10), there is a neighborhood U in G with the following property. If $G^* \supset G'$ is a finite dimensional subspace, then there is a projection of G^* into G' which carries $U \cap G^*$ into U'.

We shall relate this definition to Definition 2b of von Neumann.[21]

Note that (6) implies a separation postulate: If $g \neq 0$, then there is a U which does not contain g. The reason for using our (6) is that with it one may prove the same property, and hence the separation postulate, in tensor products.

[21] J. von Neumann, *On complete topological spaces*, Transactions of the American Mathematical Society, vol. 37(1935), pp. 1–20. We refer to this paper as N.

THEOREM 29. *A topological linear space (even with (6) replaced by a separation postulate) is a regular Hausdorff space; $g + g'$ and ag are each continuous in both variables.*

As our definition has all the properties in N, Definition 2b, except his (2) and (7), and a separation postulate holds, his proof holds without change.[22] We may now use U_{cl} = closure of U and U_i = inner points of U, etc., as in N.

Preparatory to proving Theorem 30, we note the following facts.

(a) If a set of sets U satisfies the above properties, then so do the sets U_{cl}, the sets U_i, and the sets $U - U$ (= all $g - g'$, g and g' in U).

(b) The sets U_{cl}, $(U_{cl})_i$, and $U - U$ define the same topology (i.e., give the same definition of S_i for any S) as the sets U.

These hold also if N, Definition 2b is used. To prove these facts, first note that N, Theorem 3, in particular, $(aS)_{cl} = aS_{cl}$, holds for closures; the proof is essentially the same. (a) and (b) now follow easily, using especially: $U_{cl} \subset U + U$; $V + V \subset U$ implies $V \subset U_i$.

(c) In a convex space as in N, we may suppose that the U are convex, and either closed or open, and that $-U = U$.

For we may use either the U_{cl} or the $(U_{cl})_i$. The U_{cl} are convex, by N, Theorem 13. To prove this for $(U_{cl})_i = S_i$, take g and g' in S_i and $0 < a < 1$. Set $g^* = ag + (1 - a)g'$, and choose V so that $g + V \subset S$, $g' + V \subset S$. Then

$$g^* + V \subset a(g + V) + (1 - a)(g' + V) \subset S_{cl} = S,$$

and hence g^* is in S_i. Finally, replace each U by $U - U = U'$; then all former properties hold, and $-U' = U'$.

LEMMA 6. *Let G be a convex topological linear space as in* N, *satisfying our* (c). *Let $g_1, \cdots, g_\mu, g'$, be independent; let $g_1, \cdots, g_\mu$ determine the subspace G_1 of G, and the whole set, the subspace G^*. Let m be an integer $\leqq \mu$. Let U be a neighborhood such that*

$$\text{if } \sum_{i=1}^{\mu} a_i g_i \text{ is in } U, \quad \text{then} \quad |a_i| \leqq t_i \qquad (i = 1, \cdots, m). \tag{14.1}$$

Then there is a projection of G^* into G_1 such that the projection of $U \cap G^*$ satisfies the same inequalities.

It will not restrict the generality if we suppose that t_i are the smallest numbers such that (14.1) holds. As $-U = U$, no inequality in (14.1) can be bettered now.

First, suppose we have two elements

$$g_1'' = \sum a_i g_i + cg', \qquad g_2'' = \sum b_i g_i + cg', \qquad \text{in } U; \tag{14.2}$$

then as $U = -U$ is convex,

$$\tfrac{1}{2}(g_1'' - g_2'') = \sum \tfrac{1}{2}(a_i - b_i) g_i \qquad \text{is in } U,$$

[22] In Hausdorff, *Mengenlehre*, Berlin, 1927, there is an error on p. 229: (5) does not follow from (6), as shown by a space in which the only open sets are the null set and the whole space. In N, proof of Theorem 6, one should mention that a separation postulate holds, as a consequence of Definition 2b, (2).

and hence

$$|a_i - b_i| \leqq 2t_i \qquad (i = 1, \cdots, m). \tag{14.3}$$

Now take any $c \geqq 0$ for which there is an element of the form (14.2) in U; let $\phi_i(c)$ and $\psi_i(c)$ $(i = 1, \cdots, m)$ be the greatest lower bound and least upper bound respectively of all numbers d such that for some numbers $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_\mu$,

$$\sum_{j \neq i} a_j g_j + (\pm t_i + d) g_i + c g' \qquad \text{is in } U \quad (- \text{ for } \phi, + \text{ for } \psi).$$

In other words, $\phi_i(c)$ and $\psi_i(c)$ show how much U sticks out beyond the rectangle of the t_i, in the g_i direction, at the height c, with respect to the direction of g'. By the choice of the t_i, $\phi_i(0) = \psi_i(0) = 0$.

By (14.3), $\phi_i(c) \geqq \psi_i(c)$. We now show that

$$\text{if } 0 < c < c', \text{ then } \frac{\phi_i(c)}{c} \leqq \frac{\phi_i(c')}{c'}, \qquad \frac{\psi_i(c)}{c} \geqq \frac{\psi_i(c')}{c'}. \tag{14.4}$$

Suppose, for instance, the first inequality is false. Then there are numbers a_j $(j \neq i)$, e, such that

$$g_1' = \sum_{j \neq i} a_j g_j + (-t_i + d) g_i + c' g' \qquad \text{is in } U,$$

$$d = \frac{c'}{c} [\phi_i(c) - e], \qquad e > 0.$$

By the choice of the t_i, there is an element

$$g_2' = \sum_{j \neq i} b_j g_j + (-t_i + d') g_i \text{ in } U, \qquad d' < \frac{c'e}{c' - c}.$$

As U is convex,

$$\frac{c}{c'} g_1' + \frac{c' - c}{c'} g_2' = \sum_{j \neq i} a_j' g_j + \left(-t_i + \frac{c}{c'} d + \frac{c' - c}{c'} d'\right) g_i + c g'$$

is in U. But also

$$\frac{c}{c'} d + \frac{c' - c}{c'} d' = \phi_i(c) - e + \frac{c' - c}{c'} d' < \phi_i(c),$$

contradicting the definition of $\phi_i(c)$.

The inequalities show that we may define

$$\phi_i' = \lim_{c \to 0+} \frac{\phi_i(c)}{c}, \qquad \psi_i' = \lim_{c \to 0+} \frac{\psi_i(c)}{c}; \tag{14.5}$$

then

$$\frac{\phi_i(c)}{c} \geqq \phi_i' \geqq \psi_i' \geqq \frac{\psi_i(c)}{c} \qquad (c > 0; i = 1, \cdots, m). \tag{14.6}$$

Set

$$(14.7) \qquad g'' = g' + \sum_{i=1}^{m} \phi_i' g_i ;$$

we shall show that if we project U along the direction of g'' into G_1, i.e., use $\phi(\sum a_i g_i + a''g'') = \sum a_i g_i$, then (14.1) will hold for the projection. As $-U = U$, it will be sufficient to consider the part of U with $c > 0$. If this is false, say for i, then there is an element

$$(14.8) \qquad \sum_{j=1}^{\mu} a_j g_j + cg'' \text{ in } U, \qquad c > 0, \qquad a_i > t_i \text{ or } a_i < -t_i .$$

Using (14.7), we have

$$(14.9) \qquad \sum_{j \neq i}^{j \leqq m} (a_j + c\phi_j')g_j + \sum_{j=m+1}^{\mu} a_j g_j + (a_i + c\phi_i')g_i + cg' \text{ in } U.$$

Suppose first that $a_i < -t_i$. Then

$$a_i + c\phi_i' < -t_i + c\,\frac{\phi_i(c)}{c} = -t_i + \phi_i(c),$$

contradicting the definition of $\phi_i(c)$. Next, if $a_i > t_i$, then

$$a_i + c\phi_i' > t_i + c\psi_i' \geqq t_i + \psi_i(c),$$

contradicting the definition of $\psi_i(c)$. This completes the proof.

THEOREM 30. *Any convex topological linear space as in* N *is a topological linear space as here defined, even if his* (2) *is replaced by a separation postulate.*

We may suppose his neighborhoods satisfy our (c). We must prove our (6). Let $g_1, \cdots, g_m$ form a base for G', and choose $t_1, \cdots, t_m$ so that all points $\sum a_i g_i$, $|a_i| \leqq t_i$, lie in U'. Let R be the closed region $|a_i| \leqq t_i$, and let A be its boundary. For each g in A, we may choose a $U(g)$ not containing it, and a $V(g)$ so that $V(g) - V(g) \subset U(g)$. As the operations in G are continuous, $\sum a_k g_k$ is continuous in the a_k, so the $(V(g))_i \cap G'$ (S_i = inner points of S) are open in the natural topology in G'; hence a finite number of the sets $g + V(g) \cap G'$ cover A. Let U be a neighborhood in the corresponding set $V(g_1) \cap \cdots \cap V(g_\lambda)$. Now U contains no element of A. For suppose g is in $A \cap U$. Say g is in $g_k + V(g_k)$. Then as g is in $U \subset V(g_k)$, g_k is in $V(g_k) - V(g_k) \subset U(g_k)$, a contradiction. As U is convex, $U \cap G'$ is in the complement of A in R.

Let $g_{m+1}', \cdots, g_n'$ form, with $g_1, \cdots, g_m$, a base in G^* (if $G^* \neq G'$), and let G_i be the space generated by $g_1, \cdots, g_i'$ ($i = m + 1, \cdots, n$). We shall prove, by induction on i, that G_i can be projected into G' so that $U \cap G_i$ goes into R; as $R \subset U'$, the case $i = n$ gives the theorem. There will be elements $g_{m+1}, \cdots, g_n$ such that $g_1, \cdots, g_i$ also determine G_i, and the projection of G_i into G' is with respect to $g_{m+1}, \cdots, g_i$:

$$\phi\left(\sum_{k=1}^{i} a_k g_k\right) = \sum_{k=1}^{m} a_k g_k .$$

Suppose we have found the elements $g_{m+1}, \cdots, g_\mu$. As the projection of G_μ into G' carries $U \cap G_\mu$ into R, (14.1) is satisfied. Hence we may apply Lemma 6 with $g' = g'_{\mu+1}$, $G^* = G_{\mu+1}$; this gives a projection of $G_{\mu+1}$ into G_μ such that the projection of $U \cap G_{\mu+1}$ satisfies (14.1). Let $g_{\mu+1}$ give the direction of the projection; then projecting $G_{\mu+1}$ into G' with respect to $g_{m+1}, \cdots, g_{\mu+1}$ carries $U \cap G_{\mu+1}$ into R, as required.

15. **Products, topological linear spaces.** We prove

THEOREM 31. *If G and H are topological linear spaces, so is $G \circ H$, the topology being given by* (10.12). *(We may use either open or closed neighborhoods in G and in H; see* §14.) *The multiplication $g \cdot h$ is continuous. The topology in $G \circ H$ depends only on the topologies in G and in H, not on the neighborhood systems employed.*

First, $G \circ H$ is a linear space, by Theorem 27. We shall prove the postulates of §14. (1) is trivial. To prove (2), take any two neighborhoods $N(U_1, \cdots; V_1, \cdots) = N(U_i; V_i)$, and $N(U'_i; V'_i)$. Choose U''_i in $U_i \cap U'_i$ and V''_i in $V_i \cap V'_i$; then

$$N(U''_i; V''_i) \subset N(U_i; V_i) \cap N(U'_i; V'_i).$$

To prove (3), given $N(U_i; V_i)$, choose U'_i so that $aU'_i \subset U_i$ if $|a| \leqq 1$ $(i = 1, 2, \cdots)$. Then

$$aN(U'_i; V'_i) = N(aU'_i; V'_i) \subset N(U_i; V_i).$$

To prove (4), take any $N(U_i; V_i)$. Choose U'_i and V'_i so that

$$U'_i \subset U_{2i-1} \cap U_{2i}, \qquad V'_i \subset V_{2i-1} \cap V_{2i}.$$

Now take any $\sum g_i \cdot h_i$ and $\sum g'_i \cdot h'_i$ in $N(U'_i; V'_i)$. As $g_1 \cdot h_1$ is in $U'_1 \cdot V'_1 \subset U_1 \cdot V_1$, $g'_1 \cdot h'_1$ is in $U'_1 \cdot V'_1 \subset U_2 \cdot V_2$, $g_2 \cdot h_2$ is in $U'_2 \cdot V'_2 \subset U_3 \cdot V_3$, etc., we see that $\sum g_i \cdot h_i + \sum g'_i \cdot h'_i$ is in $N(U_i; V_i)$.

To prove (5), take any $\sum_{i=1}^{s} g_i \cdot h_i$ and any $N(U_i; V_i)$. Choose U^*_i and V^*_i so that

$$cU^*_i \subset U_i, \qquad cV^*_i \subset V_i, \qquad\qquad (|c| \leqq 1; i = 1, \cdots, s),$$

take a_i and b_i so that

$$g_i \text{ is in } a_i U^*_i, \qquad h_i \text{ is in } b_i V^*_i,$$

and let a be the largest of the $|a_i|$ and $|b_i|$. Then

$$a_i U^*_i = a(a_i/a) U^*_i \subset a U_i, \qquad \text{etc.};$$

it follows that $\sum g_i \cdot h_i$ is in

$$a^2 N(U_i; V_i) = N(aU_i; aV_i).$$

We now prove (6). Let F' be a subspace of $F = G \circ H$, generated by $f_1, \cdots, f_s$. Set (see Theorem 27)

$$G' = G(f_1) + \cdots + G(f_s), \qquad H' = H(f_1) + \cdots + H(f_s).$$

Let $g_1, \cdots, g_m$ and $h_1, \cdots, h_n$ be bases in G' and H'; set $f_{ij} = g_i \cdot h_j$. Then the f_{ij} form a base (see Theorem 26) in a space $F'' \supset F'$. Take a fixed projection of F'' into F'. Given a natural neighborhood N' in F', we may choose a natural neighborhood N'' in F'' which projects into a subset of N'. As any projection of an F^* into F'' combines with the above projection to give a projection of F^* into F', it is sufficient to prove (6) with F', N' replaced by F'', N''.

Choose $\epsilon > 0$ so that any $\sum a_{ij} f_{ij}$ with each $| a_{ij} | \leqq \epsilon$ is in N''. Let A and B be the sets of elements $\sum a_i g_i$ and $\sum b_i h_i$ in G' and H' with $| a_i | \leqq \frac{1}{2}\epsilon$, $| b_i | \leqq 1$. Choose U_1 in G by (6) so that any $U_1 \cap G^*$ can be projected into A, and choose V_1 in H so that any $V_1 \cap H^*$ can be projected into B. Choose $U_2, U_3, \cdots$ so $2U_i \subset U_{i-1}$, and set $V_2 = V_3 = \cdots = V_1$. Set $N = N(U_i ; V_i)$.

Now take any $F^* \supset F''$. Choose a base $f_1^*, \cdots, f_t^*$ in F^*, and set $G^* = \sum G(f_i^*)$, $H^* = \sum H(f_i^*)$. Choose projections of G^* into G' and H^* into H' so $U_1 \cap G^*$ goes into A and $V_1 \cap H^*$ goes into B. If G_1 is the subset of G^* projecting into 0 in G', and $g_{m+1}, \cdots, g_\mu$ is a base in G_1, then $g_1, \cdots, g_\mu$ is a base in G^*; choose a base $h_1, \cdots, h_\nu$ in H^* similarly. Now any element of F^* can be written uniquely in the form

$$f = \sum_{(i,j)=(1,1)}^{(m,n)} a_{ij} f_{ij} + \sum{}' a_{ij} g_i \cdot h_j,$$

where in $\sum'$, either $i > m$ or $j > n$. (Not all such elements need be in F^*.) Dropping out the second group of terms defines a projection of F^* into F''.

We shall show by induction that any $(U_i \cdot V_i) \cap F^*$ projects into elements $\sum a_{kl} f_{kl}$ with $| a_{kl} | \leqq \epsilon/2^i$; it will follow that $N = \sum_i^* (U_i \cdot V_i)$ projects into N''.

Take first any α in $(U_1 \cdot V_1) \cap F^*$; we may suppose $\alpha \neq 0$. Then $\alpha = g \cdot h$, g in U_1, h in V_1. As α is in F^*, $G(\alpha) \subset G^*$. But also $G(\alpha) \subset G(g)$; hence $G(\alpha) \subset G^* \cap G(g)$. As $\alpha \neq 0$, $G(\alpha)$ contains elements $\neq 0$, which implies that g is in G^*. Similarly h is in H^*. Say

$$g = \sum_{i=1}^{\mu} a_i g_i, \qquad h = \sum_{j=1}^{\nu} b_j h_j.$$

Then as g projects into A and h into B, $g \cdot h$ projects into

$$\sum_{(i,j)=(1,1)}^{(m,n)} a_i b_j f_{ij}, \qquad | a_i b_j | < \tfrac{1}{2}\epsilon,$$

so that the statement holds for $(U_1 \cdot V_1) \cap F^*$. Supposing it holds for $k - 1$, we shall prove it for k. Take any g in U_k and h in V_k such that $g \cdot h$ is in F^*. Then

$$2g \text{ is in } 2U_k \subset U_{k-1}, \qquad h \text{ is in } V_{k-1},$$

so that $2(g \cdot h)$ is in $(U_{k-1} \cdot V_{k-1}) \cap F^*$, and hence projects into $\sum a_{ij} f_{ij}$ with $|a_{ij}| \leqq \epsilon/2^{k-1}$. Hence the required inequality on $g \cdot h$ holds. This completes the proof of (6).

To show that $g \cdot h$ is continuous, as $+$ is continuous in $G \circ H$ (Theorem 29), and

$$(g + g') \cdot (h + h') - g \cdot h = g \cdot h' + g' \cdot h + g' \cdot h', \tag{15.1}$$

it is sufficient to show that $g \cdot h'$ is continuous in h' at $h' = 0$, $g' \cdot h$ is continuous in g' at $g' = 0$, and $g' \cdot h'$ is continuous in g' and h' at $g' = 0$, $h' = 0$. For the first case, given $N = N(U_i\,;\,V_i)$, choose a so that g is in aU_1, and choose V so that $aV \subset V_1$ (N, Theorem 1, with $n = 1$). Then

$$g \cdot V \subset aU_1 \cdot V = U_1 \cdot aV \subset U_1 \cdot V_1 \subset N.$$

The second case is similar. The third is clear, as $U_1 \cdot V_1 \subset N$.

Finally, let $\{U\}$, $\{\bar{U}\}$ and $\{V\}$, $\{\bar{V}\}$ be equivalent neighborhood systems in G and H, respectively. Given an $N(U_i\,;\,V_i)$, choose $\bar{U}_i \subset U_i$ and $\bar{V}_i \subset V_i$ $(i = 1, 2, \cdots)$; then $\bar{N}(\bar{U}_i\,;\,\bar{V}_i) \subset N(U_i\,;\,V_i)$. Similarly find an N in any $\bar{N}$. Hence the $\{N\}$ and $\{\bar{N}\}$ are equivalent. The theorem is now completely proved.

III. Topological groups

16. **The topological tensor product.** An Abelian *topological group* G is an Abelian group which is at the same time a Hausdorff space,[23] and in which $\phi(g, g') = g + g'$ and $\psi(g) = -g$ are continuous. If $U, U', \cdots$ are the neighborhoods of 0, we may let the sets $g + U, g + U', \cdots$ be the neighborhoods of the element g, without altering the topology. If we assume that the separation postulate in Hausdorff, page 229, (4), holds, then the postulate (5) follows.

We shall say the space G is *sequence-separable* if it contains a finite or denumerable set of points forming a dense set.[24]

If G and H are sequence-separable topological groups, we define their *topological tensor product* $G \circ H$, or *tensor product* simply, as follows. Let $\bar{g}_1, \bar{g}_2, \cdots$ and $\bar{h}_1, \bar{h}_2, \cdots$ be sequences of points dense in G and H, respectively. Let $P_1, P_2, \cdots$ be a sequence of pairs of elements, $P_i = (\bar{g}_{\mu_i}, \bar{h}_{\nu_i})$, such that each pair $(\bar{g}_j, \bar{h}_k)$ occurs infinitely often among the P_i. Let T' be the discrete tensor product of G and H, with elements $\sum g_i \times h_i$. For each sequence $U_1, U_2, \cdots$ of neighborhoods (of 0) in G and each sequence $V_1, V_2, \cdots$ in H, set

$$\begin{aligned} Q'_i(U, V) &= \bar{g}_{\mu_i} \times V + U \times \bar{h}_{\nu_i} + U \times V, \\ N'(U_1, \cdots; V_1, \cdots) &= \sum_i{}^* Q'_i(U_i, V_i). \end{aligned} \tag{16.1}$$

Next, call two elements α, β of T' *equivalent*, $\alpha \sim \beta$, if every $\alpha + N$ contains β or vice versa, or if there is a succession $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$, with α_i and

[23] See Hausdorff, loc. cit. Note that neighborhoods are open sets here.

[24] For metric spaces, this is the same as separability.

α_{i+1} equivalent as above. The sets of equivalent elements form the elements of the tensor product $T = G \circ H$. The *neighborhoods* N of 0 in $G \circ H$ are the images of the sets N' in T'; they are obtained by replacing $\times$ by $\cdot$ in (16.1). *Addition* in T is the image of addition in T'. The element $\sum g_i \cdot h_i$ in T is the image of $\sum g_i \times h_i$ in T'.

THEOREM 32. *$G \circ H$ is a sequence-separable Abelian topological group; the multiplication $g \cdot h$ satisfies* (1.1), *and is continuous. The topology in $G \circ H$ is independent of the sequences $\{\bar{g}_i\}$, $\{\bar{h}_i\}$, and of the neighborhood systems $\{U\}$, $\{V\}$, employed.*

We begin by showing that T' has all the properties of a topological group, except for the separation postulate. First we prove Hausdorff's postulates (B), (C) (loc. cit., p. 228). Given $N'(U_1, \cdots; V_1, \cdots) = N'(U_l; V_l)$ and $N'(U_l'; V_l')$, take $U_i'' \subset U_i \cap U_i'$ and $V_i'' \subset V_i \cap V_i'$ $(i = 1, 2, \cdots)$; then clearly

$$N'(U_l''; V_l'') \subset N'(U_l; V_l) \cap N'(U_l'; V_l'). \tag{16.2}$$

To prove (C), it is sufficient to show that, for any $N'(U_l; V_l)$ and any α in $N'(U_l; V_l)$, there is an $N'(U_l'; V_l')$ with

$$\alpha + N'(U_l'; V_l') \subset N'(U_l; V_l). \tag{16.3}$$

As α is in $N'(U_l; V_l)$, it is in $\sum_{i=1}^{s} Q_i'(U_i; V_i)$ for some s. Choose numbers $\phi(1) > s, \phi(2) > \phi(1), \phi(3) > \phi(2), \cdots$ so that $P_i = P_{\phi(i)}$, and set $U_i' = U_{\phi(i)}$, $V_i' = V_{\phi(i)}$. Then

$$\begin{aligned} Q_i'(U_i', V_i') &= \bar{g}_{\mu_i} \times V_i' + U_i' \times \bar{h}_{\nu_i} + U_i' \times V_i' \\ &= \bar{g}_{\mu_{\phi(i)}} \times V_{\phi(i)} + U_{\phi(i)} \times \bar{h}_{\nu_{\phi(i)}} + U_{\phi(i)} \times V_{\phi(i)} = Q_{\phi(i)}'(U_{\phi(i)}, V_{\phi(i)}); \end{aligned}$$

hence $\sum_i^* Q_i'(U_i', V_i') \subset \sum_{i>s}^* Q_i'(U_i, V_i)$, and (16.3) follows.

We show that the group operations in T' are continuous. Given $N'(U_l; V_l)$, take

$$U_i' \subset U_i \cap (-U_i), \qquad V_i' \subset -V_i;$$

then

$$-Q_i'(U_i', V_i') = \bar{g}_{\mu_i} \times (-V_i') + (-U_i') \times \bar{h}_{\nu_i} + U_i' \times (-V_i') \subset Q_i'(U_i, V_i);$$

hence

$$N'(U_l'; V_l') \subset -N'(U_l; V_l), \tag{16.4}$$

and $-\alpha$ is continuous in α. To show that $\alpha + \beta$ is continuous, we must find $N'(U_l'; V_l')$ corresponding to $N'(U_l; V_l)$ such that

$$N'(U_l'; V_l') + N'(U_l'; V_l') \subset N'(U_l; V_l). \tag{16.5}$$

Choose in succession integers

$$\phi(1), \qquad \psi(1) > \phi(1), \qquad \phi(2) > \psi(1), \qquad \psi(2) > \phi(2), \qquad \cdots$$

so that $P_i = P_{\phi(i)} = P_{\psi(i)}$. Take

$$U'_i \subset U_{\phi(i)} \cap U_{\psi(i)}, \qquad V'_i \subset V_{\phi(i)} \cap V_{\psi(i)}.$$

Then as $g_{\mu_{\phi(i)}} = g_{\mu_i}$, etc.,

$$Q'_i(U'_i, V'_i) \subset Q'_{\phi(i)}(U_{\phi(i)}, V_{\phi(i)}) \cap Q'_{\psi(i)}(U_{\psi(i)}, V_{\psi(i)}).$$

Hence

$$Q'_i(\cdots) + Q'_i(\cdots) \subset Q'_{\phi(i)}(\cdots) + Q'_{\psi(i)}(\cdots),$$

and (16.5) follows.

We now consider equivalent elements in T'. First we prove

(*) *If* $\alpha \sim \beta$, *then for every* N', $\alpha + N'$ *contains* β.

For suppose there is a succession $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$, such that for each i, either every $\alpha_i + N'$ contains α_{i+1}, or every $\alpha_{i+1} + N'$ contains α_i. The latter condition implies the former. For given an N', choose $N'_1 \subset -N'$ by (16.4); then as α_i is in $\alpha_{i+1} + N'_1$, α_{i+1} is in $\alpha_i - N'_1 \subset \alpha_i + N'$. Next, given an N', choose N'_1 (using (16.5)) so that

$$N'_1 + N'_1 + \cdots + N'_1 \subset N' \qquad (n \text{ summands}).$$

Setting $A_k = N'_1 + \cdots + N'_1$ (k summands), we have

$$\alpha + N' \supset \alpha_0 + A_n \supset \alpha_1 + A_{n-1} \supset \cdots \supset \alpha_{n-1} + N'_1 \supset \beta,$$

as required.

Next we prove that T is a topological group. Let $\theta(\alpha)$ be the element of T containing the element α of T'. We must show that addition in T is uniquely defined; this is so if $\alpha \sim \alpha'$ and $\beta \sim \beta'$ imply $\alpha + \beta \sim \alpha' + \beta'$. Given any N', choose N'_1 so that $N'_1 + N'_1 \subset N'$. By the property (*), $\alpha + N'_1 \supset \alpha'$ and $\beta + N'_1 \supset \beta'$; hence

$$(\alpha + \beta) + N' \supset (\alpha + N'_1) + (\beta + N'_1) \supset \alpha' + \beta',$$

and $\alpha + \beta \sim \alpha' + \beta'$. Further,

$$\theta(\alpha + \beta) = \theta(\alpha) + \theta(\beta),$$

so that θ is a homomorphism of T' into T (which is clearly an Abelian group). To prove that T is a Hausdorff space, suppose $N_1 = \theta(N'_1)$ and $N_2 = \theta(N'_2)$ are given; take $N' \subset N'_1 \cap N'_2$; then $N = \theta(N') \subset N_1 \cap N_2$. Next, suppose α^* is in $N = \theta(N')$; then $\alpha^* = \theta(\alpha)$, α in N'. Choose N'_1 so that $\alpha + N'_1 \subset N'$; then $\alpha^* + \theta(N'_1) = \theta(\alpha + N'_1) \subset N$. To prove the separation postulate, suppose $\alpha^* \neq 0$. Say $\alpha^* = \theta(\alpha)$. As $\theta(0) = 0$, α is not ~ 0, and there is an N' not containing α. Set $N = \theta(N')$; then α^* is not in N. For if it were, then we would have $\alpha^* = \theta(\beta)$, β in N' and $\beta \sim \alpha$; but taking N'_1 by (16.3) so that $\beta + N'_1 \subset N'$, the property (*) gives $\alpha \subset \beta + N'_1 \subset N'$, a contradiction. To prove that the operations in T are continuous, given $N = \theta(N')$, take $N'_1 \subset -N'$; then $N_1 = \theta(N'_1) \subset -N$; also given $N = \theta(N')$, choose N'_1 so that $N'_1 + N'_1 \subset N'$; then $\theta(N'_1) + \theta(N'_1) \subset N$.

Next, (1) holds, as it holds for $\times$. We shall now show that $g \cdot h$ is continuous. First we show that it is continuous in h at $h = 0$; given $N = N(U_l ; V_l)$, we must find V so that $g \cdot V \subset N$. As the $\bar{g}_i$ are dense in G, we may choose j so that $g - \bar{g}_{\mu_j}$ is in U_1. Choose $V \subset V_1 \cap V_j$; then

$$g \cdot V \subset (g - \bar{g}_{\mu_j}) \cdot V + \bar{g}_{\mu_j} \cdot V \subset U_1 \cdot V_1 + \bar{g}_{\mu_j} \cdot V_1 \subset N.$$

Similarly $g \cdot h$ is continuous in g at $g = 0$. Further, $g \cdot h$ is continuous in both variables at $g = 0$, $h = 0$; for $U_1 \cdot V_1 \subset N(U_l ; V_l)$. Finally, as addition is continuous in T, (15.1) shows that $g \cdot h$ is continuous.

Next we show that T is sequence-separable; in fact, that the set of all finite sums $\sum \bar{g}_{p_i} \cdot \bar{h}_{q_i}$ is dense in T. As each element of T is a finite sum $\sum g_i \cdot h_i$ and $+$ is continuous, it is sufficient to show that for any $g \cdot h$ and any $N = N(U_l ; V_l)$ there is a $\bar{g}_i \cdot \bar{h}_j$ in $g \cdot h + N$. As $\cdot$ is continuous, we may choose U and V so that $(g + U) \cdot (h + V) \subset g \cdot h + N$; we need now merely take $\bar{g}_i$ in $g + U$ and $\bar{h}_j$ in $h + V$.

That the topology in T is independent of the neighborhood systems chosen is trivial; see the end of the proof of Theorem 31. We must still show that the topology is independent of the choice of the $\bar{g}_i$ and $\bar{h}_i$. By symmetry, it is sufficient to show that if $\{\bar{g}_i\}$ is replaced by the dense sequence $\{g_i^*\}$, then any $N(U_l; V_l)$ contains an $N^*(U_l'; V_l')$, defined in terms of the g_i^*. Let ξ_i, η_i replace μ_i, ν_i. Given $N(U_l ; V_l) = N_0$, choose $N_1, N_2, \cdots$ in succession so that $N_{i+1} + N_{i+1} \subset N_i$. As $\cdot$ and $+$ are continuous, we can choose U_i' and V_i' so that

$$Q_i^* = \bar{g}_{\xi_i} \cdot V_i' + U_i' \cdot \bar{h}_{\eta_i} + U_i' \cdot V_i' \subset N_i ;$$

then for any s,

$$\sum_{i=1}^{s} Q_i^* \subset N_1 + \cdots + N_{s-1} + N_s \subset N_1 + \cdots + N_{s-2} + N_{s-1} + N_{s-1}$$

$$\subset N_1 + \cdots + N_{s-2} + N_{s-2} \subset \cdots \subset N_1 + N_1 \subset N_0,$$

and hence $N^*(U_l'; V_l') \subset N_0$, as required. This completes the proof of the theorem.

THEOREM 33. *Let $g_1^*, g_2^*, \cdots$ and $h_1^*, h_2^*, \cdots$ be (finite or infinite) sequences such that the sets $\sum a_i g_i^*$ and $\sum a_i h_i^*$ (integral a_i) are dense in G and H, respectively. Then we may use these sequences in place of dense sequences in defining the topology in $G \circ H$.*

Let $h_1, h_2, \cdots$ be either the above sequence $h_1^*, h_2^*, \cdots$, or a dense sequence in H. Arrange all $\sum a_i g_i^*$ in a sequence $\bar{g}_1, \bar{g}_2, \cdots$. Let $N(U_l; V_l)$ be defined in terms of the sets $\{\bar{g}_i\}$, $\{h_i\}$, and $N^*(U_l; V_l)$, in terms of the sets $\{g_i^*\}$, $\{h_i\}$. It is sufficient to show that these two sets of neighborhoods give the same topology in T. As the g_i^* occur among the $\bar{g}_i$, it is clear that any N contains an N^*; we must prove the converse.

Let $P_i = (\bar{g}_{\mu_i}, h_{\nu_i})$ and $P_i^* = (g_{\xi_i}^*, h_{\eta_i})$ define the sequences of pairs defining the N and the N^*. If $\bar{g}_{\mu_i} = \sum_j a_{ij} g_j^*$, set $m(i) = \sum_j |a_{ij}|$. Then $\bar{g}_{\mu_i} \cdot V$ is contained in $m(i)$ terms of the form $g_j^* \cdot (\pm V)$, and $Q_i(U_i', V_i')$ is contained in $m(i) + 2$ terms of forms appearing (except for the $\pm$) in $N^*(U_l; V_l)$. For each i we may choose $m(i) + 2$ numbers $\phi_1(i), \cdots, \phi_{m(i)+2}(i)$ such that $\phi_k(i) \neq \phi_l(j)$ whenever $i \neq j$, and the k-th part into which Q_i is split corresponds to part of $Q^*_{\phi_k(i)}$. Then if the U_i' and V_i' are chosen small enough, we will have

$$Q_i(U_i', V_i') \subset \sum_{k=1}^{m(i)} Q^*_{\phi_k(i)}(U_{\phi_k(i)}, V_{\phi_k(i)}),$$

and hence $N(U_l'; V_l') \subset N^*(U_l; V_l)$, as required.

17. **Relation to linear spaces; examples.** We shall show that whenever the definitions of tensor products in Parts II and III both apply, they coincide.

THEOREM 34. *If G and H are sequence-separable topological linear spaces, then their topological tensor product T is the same as their topological reduced tensor product T^*.*

Let $\sum g_i \times h_i$ and $\sum g_i \cdot h_i$ denote elements of T^* and T, respectively. Set $\phi(\sum g_i \times h_i) = \sum g_i \cdot h_i$; we shall show that ϕ is a topological isomorphism. To show that ϕ is uniquely defined, we must show that $ag \cdot h = g \cdot ah$ for any real a; but this follows from the continuity of $g \cdot h$ (see §10). ϕ is a homomorphism; we shall show that it is continuous. Use $N(U_l; V_l)$ in T and $N^*(U_l; V_l)$ in T^*. Given $N(U_l; V_l)$, we wish to find $N^*(U_l'; V_l')$ mapping into it. From (10.12) and (16.1) it is apparent that we may use $U_i' = U_i$, $V_i' = V_i$.

Next we show that for any $N^* = N^*(U_l; V_l)$, there is an $N = N(U_l'; V_l') \subset \phi(N^*)$. Say $\bar{g}_1, \bar{g}_2, \cdots$ and $\bar{h}_1, \bar{h}_2, \cdots$ are the dense sequences used in G and H, and $P_1, P_2, \cdots, P_i = (\bar{g}_{\mu_i}, \bar{h}_{\nu_i})$, the pairs. By §14, (5), we may choose for each i a number a_i such that $\bar{g}_{\mu_i}$ is in $a_i U_{3i-2}$, and a number b_i such that $\bar{h}_{\nu_i}$ is in $b_i V_{3i-1}$. By von Neumann, loc. cit., Theorem 1 (with $n = 1$), there is a V_i'' such that $a_i V_i'' \subset V_{3i-2}$, and a U_i'' such that $b_i U_i'' \subset U_{3i-1}$. Choose

$$U_i' \subset U_i'' \cap U_{3i}, \qquad V_i' \subset V_i'' \cap V_{3i}.$$

Now

$$\begin{aligned}\bar{g}_{\mu_i} \cdot V_i' + U_i' \cdot \bar{h}_{\nu_i} + U_i' \cdot V_i' &\subset a_i U_{3i-2} \cdot \frac{1}{a_i} V_{3i-2} + \frac{1}{b_i} U_{3i-1} \cdot b_i V_{3i-1} + U_{3i} \cdot V_{3i} \\ &= \phi(U_{3i-2} \times V_{3i-2} + U_{3i-1} \times V_{3i-1} + U_{3i} \times V_{3i});\end{aligned}$$

hence $N \subset \phi(N^*)$.

Clearly ϕ maps T^* into the whole of T. When we have shown that ϕ is (1-1), the proof will be completed. Let T' be the discrete tensor product of G and H; use $g \circ h$ here. Take any $\alpha^* = \sum g_i \times h_i \neq 0$ in T^*; then $\alpha' = \sum g_i \circ h_i$ is a corresponding element of T'. There is an $N^* = N^*(U_l; V_l)$ which does not contain α^*. Construct $U_1', U_2', \cdots$ and $V_1', V_2', \cdots$ by the method given above, and set

$$N' = \sum_i{}^* (\bar{g}_{\mu_i} \circ V_i' + U_i' \circ \bar{h}_{\nu_i} + U_i' \circ V_i').$$

The map $\psi(\sum g_i' \circ h_i') = \sum g_i' \times h_i'$ of T' into T^* is uniquely defined. By the choice of the U_i' and V_i', $\psi(N') \subset N^*$; hence α' is not in N'. Therefore, by the property (*) in §16, α' is not ~ 0, so that the corresponding element $\sum g_i \cdot h_i$ of T is $\neq 0$. Consequently $\alpha^* \neq 0$ implies $\phi(\alpha^*) \neq 0$, and ϕ is (1-1).

Examples. That the topology in (10.12) cannot be used in the general case is shown by the example $I_0 \circ Rl$. A neighborhood U of 0 in I_0 is 0 itself; hence $U \cdot V = 0 \cdot V = 0$, and $I_0 \circ Rl$ would be discrete; the multiplication $a \cdot g$ would not be continuous. However, the sets $1 \cdot V$ form a neighborhood system in $I_0 \circ Rl$. In fact, *if G has a finite number of generators $\bar{g}_1, \cdots, \bar{g}_n$, then the sets*

(17.1) $$\bar{g}_1 \cdot V_1 + \cdots + \bar{g}_n \cdot V_n \qquad (V_1, \cdots, V_n \text{ neighborhoods in } H)$$

form a neighborhood system in $G \circ H$. This is an easy consequence of Theorem 33 (compare Theorem 26).

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