Trivial cofibration-fibration factorization with one application

Introduction

This note contains two results.

We first describe a trivial cofibration-fibration factorisation of a map between cubical set as an inductive definition (for *cartesian* cubical sets). Using this decomposition, and the results in Section 2 of [1], it is then possible to define a a model structure (the *type-theoretic* model structure) on cartesian cubical sets. The fibrant objects for this model structure form a model of dependent type theory with univalence, and this is most directly seen using internal language of a presheaf model.

We also have a canonical notion of geometric realization of a cubical set, where the formal interval $\mathbb{I}$ is interpreted by the unit interval $[0,1]$, and it is natural to ask if this geoemtric realization defines an equivalence of Quillen model structure. The second part of this note describes a negative result of Christian Sattler. Our presentation uses the explicit description of the trivial cofibration-fibration given in the first part. If we define Q to be the *generic* symmetric square, then the geometric realization of Q should be like a triangle, and so should be contractible. We explain however (following Sattler) that the canonical map $Q \rightarrow 1$ cannot be an equivalence for the type-theoretic model structure.

1 Notations

The objects of the base category $I, J, K, \ldots$ are finite set of names (disjoint from $0,1$) written $x, y, z, \ldots$. We supposed that we have a fresh name function on names and write I^+ the set I where we have added a fresh name. A map $f : J \rightarrow I$ is a set theoretic function $f : I \rightarrow J \cup \{0,1\}$. We have a projection map $\mathsf{p} : J^+ \rightarrow J$ which corresponds to the inclusion $J \rightarrow J \cup \{0,1\}$.

If $f : J \rightarrow I$ we define in a functorial way $f^+ : J^+ \rightarrow I^+$.

The formal interval is defined by $\mathbb{I}(J) = J \cup \{0,1\}$.

We let $\Omega(J)$ to be the set of *decidable* sieves on I.

If k is in $\mathbb{I}(J)$ we have a map $[k] : J \rightarrow J^+$ which sends the fresh name to k. We have $\mathsf{p}[k] = 1_J$.

A square in a cubical set can be written as an object $a(x,y)$ which depends of two names x and y. The edges of this square are then $a(0,0), a(0,1), a(1,0)$ and $a(1,1)$ and the diagonal (in the direction of name z) is $a(z,z)$.

2 Fibrations

Given a cubical set B, a "dependent presheaf E over B is given by a presheaf on the category of elements of B. It is thus given by a family of sets $E(v)$ for v in $B(I)$ with restriction maps $u \mapsto uf,\ E(v) \rightarrow E(vf)$ for $f : J \rightarrow I$. We can then define the total space $\Sigma\ B\ E$ and the projection map $\Sigma\ B\ E \rightarrow B$.

We express when this projection map is a *fibration*.

We first present the definition using freely the internal language of presheaf models. In this language we see E as a dependent family $E(v)$ for $v : B$. We also have a family of subsingletons $[\psi] = \{\mathsf{tt} \mid \psi\}$ for $\psi : \Omega$. We define E to be a *fibration* iff there is an operation f_E which given $\gamma : B^{\mathbb{I}}$ and $k : \mathbb{I}$ and a partial section

$$v : \Pi(i : \mathbb{I})[\psi \vee i = k] \rightarrow E\gamma(i)$$

builds a total section $f_E v : \Pi(i : \mathbb{I}) E\gamma(i)$ such that $f_E v$ extends v, i.e. $\psi \vee i = k \Rightarrow f_E\ v\ i = v\ i\ \mathsf{tt}$.

This expresses the right lifting property w.r.t. generating trivial cofibrations, with an explicit lifting operation.

A simpler notion, which is equivalent, is to have a *composition* operation. Given $\gamma : B^{\mathbb{I}}$ and $k, l : \mathbb{I}$ and a partial section

$$v : \Pi(i : \mathbb{I})[\psi \vee i = k] \to E\gamma(i)$$

the operation $c_E^{k\to l} v$ builds an element in $E\gamma(l)$ such that $\psi \Rightarrow c_E^{k\to l} v = v\ l\ \mathsf{tt}$ and $c_E^{k\to k} v = v\ k\ \mathsf{tt}$.

Given a composition operation c_E we can define a filling operation $f_E\ v = \lambda(i : \mathbb{I}) c_E^{k\to i}\ v$.

Let us unfold these definitions in the presheaf model.

Given v in $B(J^+)$ (this corresponds to γ) and k in $\mathbb{I}(J)$ and u_k in $E(v[k])$ and ψ in $\Omega(J)$ and a family of elements u_f in $E(vf)$ for $f : K \to J^+$ satisfying $\psi \mathsf{p} f = 1$ and such that $u_f g = u_{fg}$ if $g : L \to K$ and $u_{[k]g} = u_k g$, for $g : K \to J$ such that $\psi g = 1$, we should find a *filling*, that is an element $f_E\ u_k\ (\psi, u)$ in $E(v)$ such that $(f_E\ u_k\ (\psi, u))[k] = u_k$ and $(f_E\ u_k\ (\psi, u))f = u_f$ whenever $\psi \mathsf{p} f = 1$. Furthermore $(f_E\ u_k\ (\psi, u))g^+ = f_E\ u_k g\ (\psi g, u g^+)$ if $\psi g \neq 1$. The pair $u_k, (\psi, u)$ corresponds to the partial section in the internal definition.

The composition operation, under the same hypotheses, and given another element $l \neq k$ in $\mathbb{I}(J)$, is required to find an element $\mathsf{comp}^{k\to l}(u_k, \psi, u) = u_l$ in $E(v[l])$ such that $u_l g = u_{[l]g}$ if $g : K \to J$ satisfies $\psi g = 1$ and $u_l g = u_k g$ if $kg = lg$. (Note that we have $u_k g = u_{[k]g} = u_{[l]g} = u_l g$ if both conditions $\psi g = 1$ and $kg = lg$ hold.) Furthermore $\mathsf{comp}^{k\to l}(u_k, \psi, u)g = \mathsf{comp}^{kg\to kl}(u_k g, \psi g, u g^+)$ if $kg \neq lg$ and $\psi g \neq 1$.

For the trivial cofibration-fibration factorization, one intuition is that we "force" a map $\sigma : A \to B$ to become a fibration by adding in a "free" way (with constructor) a composition operation.

3 Trivial cofibration-fibration factorization

Let $\sigma : A \to B$ be a map of cubical sets. We explain how to build a trivial cofibration-fibration factorization of this map.

We first define a family of sets $F(v)$ for v in $B(J)$ together with maps $F(v) \to F(vf)$ for $f : J \to I$.

(This will be an upper approximation of a dependent type E over B and the desired factorization will be of the form $A \to \Sigma\ B\ E \to B$.)

This is defined inductively:

- $\mathsf{inc}\ a$ in $F(\sigma\ a)$ if a is in $A(I)$
- $\mathsf{comp}^{k\to l}(u_k, \psi, u, v)$ in $F(v[l])$ if u_k in $F(v[k])$ and $k \neq l$ in $\mathbb{I}(J)$ and v in $B(J^+)$ and $\psi \neq 1$ in $\Omega(J)$ and u is a family of elements u_f in $F(vf)$ for $f : K \to J^+$ such that $\psi p f = 1$.

We then define, for $g : L \to J$ and w in $B(J)$ and u in $E(w)$ an element ug in $E(wg)$:

- $(\mathsf{inc}\ a)g = \mathsf{inc}\ (ag)$
- $(\mathsf{comp}^{k\to l}(u_k, \psi, u, v))g = \mathsf{comp}^{kg\to lg}(u_k g, \psi g, u g^+, v g^+)$ if $kg \neq lg$ and $\psi g \neq 1$ where $(ug^+)_h = u_{g^+h}$ if $g : K \to J$ and $h : L \to K^+$
- $(\mathsf{comp}^{k\to l}(u_k, \psi, u, v))g = u_k g$ if $kg = lg$
- $(\mathsf{comp}^{k\to l}(u_k, \psi, u, v))g = u_{[l]g}$ if $\psi g = 1$

Note that the two last cases may happen at the same time: we can have $kg = lg$ *and* $\psi g = 1$ but we fix then the result to be $u_k g$.

We define now inductively a subset $E(v)$ of $F(v)$

- $\mathsf{inc}\ a$ is in $E(\sigma\ a)$
- $\mathsf{comp}^{k\to l}(u_k, \psi, u, v)$ is in $E(v[l])$ provided u_k is in $E(v[k])$ and u_f is in $E(vf)$ for $f : K \to J^+$ such that $\psi p f = 1$ and $u_{[k]g} = u_k g$ if $g : K \to J$ such that $\psi g = 1$ and $u_f g = u_{fg}$ if $g : L \to K$.

It can then be checked that if u is in $E(v)$ and $g : K \to J$ then ug is in $E(vg)$ and we have $(ug)h = u(gh)$ in $E(vgh)$ if $h : L \to K$.

In this way, we define a fibrant type family E over B, of total space T, and we have a factorization $A \to T,\ a \mapsto (\sigma\ a, \mathsf{inc}\ a)$ and $T \to B,\ (v, u) \mapsto v$ of the given map $\sigma : A \to B$.

It is possible to check that $A \to T$ has the left lifting property w.r.t. any fibration.

It would actually be possible to take the direct definition of E as primitive. (This is what we did in our cubical type theory paper.) One would define directly $E(v)$ by the clauses:

- $\mathsf{inc}\ a$ in $E(\sigma\ a)$ if a is in $A(I)$
- $\mathsf{comp}^{k \to l}(u_k, \psi, u, v)$ in $E(v[l])$ if u_k in $E(v[k])$ and $k \neq l$ in $\mathbb{I}(J)$ and v in $B(J^+)$ and $\psi \neq 1$ in $\Omega(J)$ and u is a family of elements u_f in $F(vf)$ for $f : K \to J^+$ such that $\psi pf = 1$, with the conditions $u_k g = u_{[k]g}$ and $u_f g = u_{fg}$ if $g : L \to K$

and at the same time

- $(\mathsf{inc}\ a)g = \mathsf{inc}\ (ag)$
- $(\mathsf{comp}^{k \to l}(u_k, \psi, u, v))g = \mathsf{comp}^{kg \to lg}(u_k g, \psi g, ug^+, vg^+)$ if $kg \neq lg$ and $\psi g \neq 1$ where $(ug^+)_h = u_{g^+ h}$ if $g : K \to J$ and $h : L \to K^+$
- $(\mathsf{comp}^{k \to l}(u_k, \psi, u, v))g = u_k g$ if $kg = lg$
- $(\mathsf{comp}^{k \to l}(u_k, \psi, u, v))g = u_{[l]g}$ if $\psi g = 1$

Note that there is no ambiguity in the last two cases, since we have $u_k g = u_{[k]g} = u_{[l]g}$ if $kg = lg$ and $\psi g = 1$.

This definition looks similar to an inductive-recursive definition: we define at the same time the sets $E(v)$ and the restriction maps $E(v) \to E(vg)$.

4 A property of cubical sets

We say that a cube $c(x, y, z)$ *connects* a square $a(x, y)$ to a square $b(x, y)$ if we have $c(x, y, 0) = a(x, y)$ and $c(x, y, 1) = b(x, y)$ *or* $c(x, y, 1) = a(x, y)$ and $c(x, y, 0) = b(x, y)$. (This defines a homotopy between a and b.)

We consider the following property of cubical sets, written $P(A)$: if we have a square $a(x, y)$ in $A(x, y)$ such that $a(x, y) = a(y, x)$ then we can find a sequence of *cubes* $c_k(x, y, z)$ which connects $a(x, y)$ to a constant square in a symmetric way, i.e. $c_k(x, y, z) = c_k(y, x, z)$ for all k.

Lemma 4.1 *If $u : A \to B$ has a section then $P(A)$ implies $P(B)$.*

Proof. If v is a section of u and $b(x, y) = b(y, x)$ is a symmetric square in $B(x, y)$ then vb is a symmetric square in $A(x, y)$ and we have a sequence of homotopies in A which can be mapped by u. □

Proposition 4.2 *If $\sigma : A \to B$ is an equivalence then $P(A)$ implies $P(B)$.*

Proof. σ being an equivalence means by definition that if we consider a trivial cofibration-fibration factorisation $A \to T \to B$ of σ then the fibration $T \to B$ is a *trivial* fibration.

We Have given above a concrete description of T and the map $A \to T$ where T is described as a total space of a dependent family $B \vdash E$.

$E(v)$ is defined inductively as follows: an element of $E(v)$ for v in $E(J)$ is either

- $\mathsf{inc}\ a$, a in $A(J)$ in $E(\sigma\ a)$
- $\mathsf{comp}^{k \to l}(u_k, \varphi, u, v)$ in $E(v[l])$ with $k \neq l$ in $\mathbb{I}(J)$ with u_k in $E(v[k])$ and $\varphi \neq 1$ in $\Omega(J)$ and u partial element of extent φ (the exact definition will not matter here) and v in $E(J^+)$

We prove that we always have $P(A)$ implies $P(T)$ (even if σ is not an equivalence).

Indeed a symmetric square in T is

- either of the form $(\sigma\ a, \mathsf{inc}\ a)$ where a is a symmetric square in A, and we can conclude using $P(A)$
- or of the form $(v(x,y,l), \mathsf{comp}^{k \to l}(a, \varphi, u, \langle z \rangle v(x,y,z)))$ with $k \neq l$ in $\mathbb{I}(x,y)$. This is symmetric in x, y so we should have $(k,l) = (0,1)$ or $(1,0)$ and $a(x,y) = a(y,x)$ and $v(x,y,z) = v(y,x,z)$. This square is then homotopic to $(v(x,y,k), a)$ by a cube symmetric in x, y, namely $(v(x,y,t), \mathsf{comp}^{k \to t}(a, \varphi, u, \langle z \rangle v(x,y,z)))$ and we can conclude by induction. (We write z the fresh name for x, y and it is bound by the operation comp.)

So we have $P(A)$ implies $P(T)$.

Note that the homotopy can go from 0 to 1 (if $k = 0$) or from 1 to 0 (if $k = 1$) and we work with non necessarily fibrant cubical sets.

If now $T \to B$ is a *trivial* fibration, then it has a (strict) section and we conclude by the previous Lemma. □

5 Application

Let now Q be the quotient of $Yon(x,y)$ by swapping x and y. Concretely Q can be seen as a nominal set with only one primitive symmetric square $q(x,y)$.

The set $Q()$ has 3 elements $q(0,0), q(1,1)$ and $q(1,0) = q(0,1)$.

The set $Q(x)$ contains $Q()$ and has also for elements $q(0,x) = q(x,0), q(1,x) = q(x,1), q(x,x)$.

In general the set $Q(J)$ has for elements $q(i,j) = q(j,i)$ where i and j vary over $0, 1$ and the elements of J.

If $Q \to 1$ is an equivalence, then, by 3 out of 2, any global point $1 \to Q$ (for instance $q(0,0)$) is also an equivalence. Since we trivially have $P(1)$, we should also have $P(Q)$ by the Proposition.

But any cube $b(x,y,z)$ in Q such that $b(x,y,0) = q(x,y)$ has to be $b(x,y,z) = q(x,y)$ and so it is constant in z and so we *cannot* have $P(Q)$.

So $Q \to 1$ is not a weak equivalence.

Remark 1: if we have connections we have $P(Q)$ since we can define $b(x,y,z) = a(x \vee z, y \vee z)$ such that $b(x,y,0) = a(x,y)$ and $b(x,y,1) = a(1,1)$ and $b(x,y,z) = b(y,x,z)$.

Remark 2: it looks like the Lemma is also valid for cubical sets with connections.

Remark 3: the Lemma should also be valid for de Morgan algebra cubes but where we replace the symmetric square by the symmetric line L containing a line $l(x) = l(1-x)$: this line L is not equivalent to 1.

Remark 4: it follows from this result that the fibrant replacement of Q is *not* contractible. It is possible to describe the fibrant replacement of $Q \to R(Q)$ in a direct combinatorial way. It follows from the argument that we dont have any homotopy $Q \times \mathbb{I} \to R(Q)$ which connects the generic symmetric square $q(x,y)$ to a constant square. But though this is a purely combinatorial result, it is not easy to see directly why it holds.

References

[1] Ch. Sattler. The equivalence extension property and model structures https://arxiv.org/pdf/1704.06911