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Professor Nicola Gambino  
University of Leeds

Professor Marcelo Fiore  
University of Cambridge

This is a letter in support of the application of Dr Karol Szumilo to a Research Fellowship in Pure Mathematics (EPSMA1027) at the University of Leeds.

Dr Szumilo received his PhD from the University of Bonn in 2014, with a thesis in algebraic topology under the direction of Professor Stefan Schwede. He subsequently got a post-doctoral Fellowship at the University of Western Ontario in Canada, and a Research Fellowship at the University of Leeds.

Dr Szumilo has two papers from his thesis "*Homotopy theory of cofibrations categories*" and "*Frames in cofibrations categories*". The main theorem states that the homotopy theory of Brown cofibration categories is equivalent to the homotopy theory of finitely cocomplete quasi-categories. The theorem is actually more general: the homotopy theory of  $\kappa$ -cocomplete Brown cofibration categories is equivalent to the homotopy theory of  $\kappa$ -cocomplete quasi-categories for any regular cardinal  $\kappa$ . The proof depends on showing that the category  $\mathbf{CofCat}_\kappa$  of  $\kappa$ -cocomplete Brown cofibration categories and  $\kappa$ -cocontinuous functors has itself the structure of a Brown fibration category and similarly for the category  $\mathbf{QCat}_\kappa$  of  $\kappa$ -cocomplete quasi-categories and  $\kappa$ -cocontinuous functors. Dr Szumilo constructs two functors in opposite directions:

$$\begin{array}{ccc} & N_f & \\ \text{CofCat}_\kappa & \xrightarrow{\quad} & \text{QCat}_\kappa \\ & Dg & \end{array}$$

The functor  $N_f$  associates to a cofibration category  $\mathcal{C}$  its category of *frames*  $N_f(\mathcal{C})$ ; the construction is a homotopical version of Gabriel-Zisman calculus of fractions. The functor  $Dg$  associates to a  $\kappa$ -cocomplete quasi-category  $\mathcal{X}$  its category of  $\kappa$ -small diagrams  $Dg(\mathcal{X})$ . The functors  $N_f$  and  $Dg$  are exact and mutually homotopy inverse.

The theory of Dr Szumilo has applications to the theory of  $C^*$ -algebras. In a paper collaboration with Land and Nikolaus, "*Localisation of cofibration categories and groupoid  $C^*$ -algebras*", the authors show that if a cofibration category  $\mathcal{C}$  has a good cylinder, then its Dwyer-Kan localisation is equivalent to

the Dwyer-Kan localisation of its subcategory of cofibrations  $c\mathcal{C}$ . The result is proved by showing that the inclusion functor  $c\mathcal{C} \rightarrow \mathcal{C}$  induces a levelwise weak homotopy equivalence of classification diagrams  $\mathbf{N}(c\mathcal{C}) \rightarrow \mathbf{N}(\mathcal{C})$ . The result has applications to enveloping  $C^*$ -algebras of countable groups.

In a paper in collaboration with Professor K. Kapulkin " *Quasicategories of frames of cofibration categories*", the authors show that the quasi-category of frames  $N_f(\mathcal{C})$  of a cofibration category  $\mathcal{C}$  is a Dwyer-Kan localisation of  $\mathcal{C}$ . The proof depends on Rezk's construction the Dwyer-Kan localisation of a relative category. The main step is to show that the classification diagram  $\mathbf{N}(\mathcal{C})$  of  $\mathcal{C}$  is Rezk equivalent to a complete Segal space whose first row is isomorphic to the quasi-category  $N_f(\mathcal{C})$ . The main result has applications to Homotopy Type Theory. In addition to that, the authors construct two models of the Dwyer-Kan localisation of a Quillen model  $\mathcal{E}$ . Let  $\mathcal{E}_c$  be the full subcategory of cofibrant objects of  $\mathcal{E}$ , and let  $\mathcal{E}_f$  be the full subcategory of fibrant objects of  $\mathcal{E}$ ; the category  $\mathcal{E}_c$  has the structure of a cofibration category, whereas  $\mathcal{E}_f$  has the structure of a fibration category; hence the quasi-category  $N_f(\mathcal{E}_c)$  is finitely cocomplete and the quasi-category  $N_f^{op}(\mathcal{E}_f) := N_f((\mathcal{E}_f)^{op})^{op}$  is finitely complete. The authors constructs a third quasi-category  $N_f(\mathcal{E})$  together with two categorical equivalences

$$N_f(\mathcal{E}_c) \leftarrow N_f(\mathcal{E}) \rightarrow N_f^{op}(\mathcal{E}_f).$$

It shows that the quasi-categories  $N_f(\mathcal{E}_c)$  and  $N_f^{op}(\mathcal{E}_f)$  are equivalent and hence that they are both finitely complete and cocomplete.

In a paper in collaboration with K. Kapulkin, " *Internal Language of Finitely Complete  $(\infty, 1)$ -categories*", the authors show that the homotopy theory of fibration categories is equivalent to the homotopy theory of tribes. The notion of tribe (initially introduced by M. Shulman) is a categorical formulation of dependant type theory with identity types and sums. The main theorem of the paper states that the forgetful functor  $U : \mathbf{Trb} \rightarrow \mathbf{FibCat}$ , from the category of tribes to the category of fibration categories is a Dwyer-Kan equivalence. For the proof, the authors introduce a semi-simplicial version of the notions of tribe and of fibration category. They show that the two vertical functors in the following square

$$\begin{array}{ccc} s\mathbf{Trb} & \xrightarrow{U'} & s\mathbf{FibCat} \\ \downarrow & & \downarrow \\ \mathbf{Trb} & \xrightarrow{U} & \mathbf{FibCat} \end{array}$$

are *DK*-equivalences. They also show that the functor  $U'$  is a *DK*-equivalences by showing that it has the Waldhausen approximation property. It follows that the functor  $U$  is a *DK*-equivalence. As an application, they show that the category of comprehension categories with identity types and  $\Sigma$ -types is *DK*-equivalent to the category of finitely complete quasi-categories. This means that an important fragment of type theory can be fully interpreted in  $(\infty, 1)$ -categories with finite limits. The paper is a master-piece of homotopical algebra.

Dr Szumilo has a recent paper in collaboration with N. Gambino and C. Sattler *On the constructive Kan-Quillen model structure: two new proof*. In addition to showing that Henry's work can be greatly streamlined, the paper is opening a new perspective: it appears possible to extend many results to the category of simplicial objects  $s\mathcal{E}$  of a Grothendieck topos  $\mathcal{E}$ . Dr Szumilo visited me September 2019 to discuss his work. I wondered if the constructive model structure on  $s\mathcal{E}$  could be the homotopy theoretic analog of the left exact completion of  $\mathcal{E}$ . I am happy to see that this is now answered in the draft of a future paper *The effective model structure and  $\infty$ -groupoids* in collaboration with N. Gambino, S. Henry and C. Sattler. The authors show that under mild hypothesis on a category  $\mathcal{E}$ , the category  $s\mathcal{E}$  admits a constructive model structure, called the *effective model structure*. Surprisingly, the model category  $s\mathcal{E}$  satisfies Rezk descent condition of an  $\infty$ -topos! It may thus model homotopy type theory! Also, the authors show that the  $\infty$ -category presented by the model category  $s\mathcal{E}$  is equivalent to the full subcategory of presheaves on  $\mathcal{E}$  spanned by the Kan complexes in  $\mathcal{E}$ . This is showing that the model category  $s\mathcal{E}$  is the homotopy theoretic analog of the left exact completion of  $\mathcal{E}$ . The authors are aware that Hyland's *effective topos* is the left exact completion of the category of recursive families of non-empty sets. It is therefore natural to ask the question: is there an homotopical version of Hyland's effective topos, the *effective  $\infty$ -topos*? A positive answer would bring the powerful methods of recursive realizability to homotopy type theory.

Dr Karol Szumilo is a very talented young mathematician. He has developed the link between the theory of cofibration categories and that of quasi-categories. He has developed an internal language for category theory within type theory. He is having important contributions to constructive simplicial homotopy theory. He could contribute greatly to a project for constructing the effective  $\infty$ -topos. I recommend him strongly and enthusiastically to the Research Fellowship in Pure Mathematics at the University of Leeds.

Sincerely yours,

André Joyal  
 Professor Emeritus in Mathematics, UQAM  
 Fellow of the Royal Society of Canada

joyal.andre@uqam.ca